Extremal Hypergraph Theory and Algorithmic Regularity Lemma for Sparse Graphs

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Abstract

Once invented as an auxiliary lemma for Szemerédi's Theorem [106] the regularity lemma [105] has become one of the most powerful tools in graph theory in the last three decades which has been widely applied in several fields of mathematics and theoretical computer science.

Roughly speaking the lemma asserts that dense graphs can be approximated by a constant number of bipartite quasi-random graphs, thus, it narrows the gap between deterministic and random graphs. Since the latter are much easier to handle this additional information is often very useful.

With Szemerédi's regularity lemma as the starting point two roads diverge in this thesis aiming at applications of the concept of regularity on the one hand and clarification of several aspects of this concept on the other.

In the first part we deal with questions from extremal hypergraph theory and foremost we will use the so-called weak regularity lemma for uniform hypergraphs, a generalised version of Szemerédi's regularity lemma, to prove asymptotically sharp bounds on the minimum degree which ensure the existence of Hamilton cycles in uniform hypergraphs. Moreover, we derive (asymptotically sharp) bounds on minimum degrees of uniform hypergraphs which guarantee the appearance of perfect and nearly perfect matchings.

In the second part a novel notion of regularity will be introduced which generalises Szemerédi's original concept. Concerning this new concept we provide a polynomial time algorithm which computes a regular partition for given graphs without too dense induced subgraphs. This generalises the result of Alon, Duke, Lefmann, Rödl and Yuster on algorithmic regularity lemma [9] as well as the result of Kohayakawa on regularity lemma for sparse graphs [61]. As an application we show that for the above mentioned class of graphs the problem MAX-CUT can be approximated within a multiplicative factor of $(1 + o(1))$ in polynomial time.

Furthermore, pursuing the line of research of Chung, Graham and Wilson [22, 17, 21] on quasi-random graphs we study the notion of quasi-randomness resulting from the new notion of regularity and concerning this we provide a characterisation in terms of eigenvalue separation of the normalised Laplacian matrix.

Zusammenfassung

Einst als Hilfssatz für Szemerédis Theorem [106] entwickelt, hat sich das Regularitätslemma [105] in den vergangenen drei Jahrzehnten als eines der wichtigsten Werkzeuge der Graphentheorie etabliert und breite Anwendung in vielen Bereichen der Mathematik und der Theoretischen Informatik gefunden.

Im Wesentlichen hat das Lemma zum Inhalt, dass dichte Graphen durch eine konstante Anzahl quasizufälliger, bipartiter Graphen approximiert werden können, wodurch zwischen deterministischen und zufälligen Graphen eine Brücke geschlagen wird. Da letztere viel einfacher zu handhaben sind, stellt diese Verbindung oftmals eine wertvolle Zusatzinformation dar.

Vom Regularitätslemma Szemerédis ausgehend gliedert sich die vorliegende Arbeit in zwei Teile, die zum einen Gebrauch vom Konzept der Regularität machen und zum anderen verschiedene Aspekte dieses Begriffs beleuchten.

Mit Fragestellungen der Extremalen Hypergraphentheorie beschäftigt sich der erste Teil der Arbeit. Es wird zunächst die für Hypergraphen verallgemeinerte Version des Regularitätslemmas angewandt, um asymptotisch scharfe Schranken für das Auftreten von Hamiltonkreisen in uniformen Hypergraphen mit hohem Minimalgrad herzuleiten. Nachgewiesen werden des Weiteren asymptotisch scharfe Schranken für die Existenz von perfekten und nahezu perfekten Matchings in uniformen Hypergraphen mit hohem Minimalgrad.

Im zweiten Teil der Arbeit wird ein neuer, Szemerédis ursprüngliches Konzept generalisierender Regularitätsbegriff eingeführt. Diesbezüglich wird ein Algorithmus vorgestellt, welcher zu einem gegebenen Graphen ohne zu dichte induzierte Subgraphen eine reguläre Partition in polynomieller Zeit berechnet. Sowohl das Resultat von Alon, Duke, Lefmann, Rödl und Yuster zu algorithmischem Regularitätslemma [9], als auch jenes von Kohayakawa zu Regularitätslemma für dünne Graphen [61] werden damit verallgemeinert. Als eine Anwendung dieses Resultats wird darüber hinaus gezeigt, dass das Problem MAX-CUT für die oben genannte Graphenklasse in polynomieller Zeit bis auf einen multiplikativen Faktor von $(1 + o(1))$ approximierbar ist.

Der Untersuchung von Chung, Graham und Wilson [22, 17, 21] zu quasizufälligen Graphen folgend wird ferner der sich aus dem neuen Regularitätskonzept ergebende Begriff der Quasizufälligkeit studiert und in Hinsicht darauf eine Charakterisierung mittels Eigenwertseparation der normalisierten Laplaceschen Matrix angegeben.

Contents

1. Prologue

1.1. Szemerédi's theorem and the regularity lemma for graphs

The starting point of this thesis is Szemerédi's regularity lemma for graphs. First invented by Szemerédi in 1975 it has become an important tool in various fields, including graph theory, combinatorial number theory, combinatorial geometry and theoretical computer science.

Arithmetic progressions in the integers The first form of this lemma appeared in the seminal paper of Szemerédi [106] and served as a crucial tool in resolving the famous conjecture of Erdős and Turán [28] on the upper density of subsets of integers containing no arithmetic progressions.

Theorem 1 (Szemerédi '75)**.** *For all $k \in \mathbb{N}$ and $\delta > 0$ there exists an n_0, such that for every $n > n_0$ the following holds. Let A be a subset of $[n] = \{1, 2, \ldots, n\}$ which satisfies $|A| > \delta n$, then A contains an arithmetic progression of length k, i.e., there exist numbers a and $\ell \neq 0$ such that $\{a, a + \ell, \ldots, a + (k-1)\ell\} \subset A$.*

This "masterpiece of combinatorial reasoning" [48] has set the stone rolling. Since then many efforts have been dedicated to find alternative approaches to Theorem 1 which has led to various proofs with very different types of analysis. This in turn provided many insights into various branches of mathematics and also showed many connections between them.

Furstenberg's ergodic theoretical approach [37] to this problem, e.g., has found several extensions [38, 39, 40], including the proof of the density version of the Hales-Jewett theorem [52]. The Fourier analytical approach, first introduced by Roth [98] (proving Theorem 1 for $k = 3$), resulted in Gowers' proof [44, 45] of Theorem 1 with the currently best bound on n_0 which is doubly exponential in $\mathrm{poly}(1/\delta)$ (Szemerédi's proof yielded a tower type bound for n_0). The combinatorial approach to Szemerédi's theorem has been extended by Gowers [46, 47] and by Nagel, Rödl, Schacht, and Skokan [88, 91, 92] which has led to a much better understanding of the regularity lemmas for hypergraphs. Many tools developed in this field, including Szemerédi's regularity lemma, have been widely and successfully used in theoretical computer science and we refer to [111] for further reading.

Szemerédi's regularity lemma The heart of Szemerédi's combinatorial approach to Theorem 1 is the regularity lemma for dense graphs. Roughly speaking, the lemma states that dense graphs can be approximated by a constant number of "random-like" bipartite graphs. This allows one to carry over probabilistic intuition and reasoning from the theory of random graphs into deterministic settings. Since random graphs are usually much easier to handle than arbitrary graphs with the same density, this additional information is often very valuable.

The above-mentioned similarity to random bipartite graphs is captured by the concept of regularity which measures the edge distribution of the bipartite graph. Let $G = (V, E)$ be a graph and for two disjoint subsets $X, Y \subset V$ let $e(X, Y)$ denote the number of edges with one end in X and the other in Y. Further, let

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}$$

denote the **density** of the pair (X, Y) and we say that (X, Y) is quasi-random with parameter ϵ, in short **ϵ-regular**, if

$$|d(A, B) - d(X, Y)| < \epsilon \tag{1.1}$$

for all $A \subset X$ and $B \subset Y$ satisfying $|A| > \epsilon|X|$ and $|B| > \epsilon|Y|$. Roughly speaking, this asserts that the edge density of every large enough subpair of the pair (X, Y) is close to the edge density of (X, Y) itself (with ϵ governing the precision of approximation), i.e. the edges of the pair (X, Y) are "smoothly" distributed. This is a "deterministic" property which nevertheless reflects a characteristic property of many natural models of random bipartite graphs[1]. This legitimises the notion of quasi-randomness and in the following we may use the two words, regularity and quasi-random, interchangeably. (At the same time we also want to warn the reader that we may use the word "quasi-random" in several places without referring to a particular, rigid definition but take it as an allusion to a more general phenomenon which will hopefully be clear from the context.)

Observe that (1.1) is void for sparse pairs, i.e. pairs (X, Y) with $o(|X||Y|)$ edges, since $d(A, B)$ and $d(X, Y)$ both tend to zero. Hence, for sparse pairs, the property (1.1) does not yield any additional information. In contrast, for dense pairs, i.e. pairs (X, Y) having $\Omega(|X||Y|)$ edges, the control is quite striking, as seen, e.g., by the following.

Theorem 2 (Counting lemma)**.** *For every $k \in \mathbb{N}$, $d_0, \delta > 0$ there exists an $\epsilon > 0$ and n_0 such that the following holds. Let $(V_1, \ldots, V_k)$ with $|V_i| = n \geq n_0$ be the partition classes of a k-partite graph G and for any $1 \leq i < j \leq k$ suppose the pair (V_i, V_j) is ϵ-regular and has density $d > d_0$. Then G contains $d^{\binom{k}{2}} n^k \pm \delta n^k$ copies of K_k, the complete graph on k vertices.*

Note that $d^{\binom{k}{2}} n^k$ is what we would expect from the random k-partite graph with density d between each pair and by $\pm x$ we mean a quantity lying between $-x$ and x.

Bridging the gap of quasi-random graphs and dense graphs, Szemerédi proved the following.

[1] e.g. the binomial model where each possible edge appears with probability $p \in (0, 1)$ independently

Theorem 3 (Szemerédi's regularity lemma)**.** *For all $\epsilon > 0$ and t_0 there exist a T_0 and an n_0, such that for all graphs $G = (V, E)$ on $n > n_0$ vertices there exists a partition of V into $V_0, V_1, \ldots, V_t$, which satisfies the following properties:*

- $t_0 < t < T_0$,
- $|V_0| < \epsilon n$,
- $|V_1| = |V_2| = \cdots = |V_t|$ *and*
- *all but at most $\epsilon\binom{t}{2}$ pairs (V_i, V_j) , $1 \leq i < j \leq t$, are ϵ-regular.*

Beyond its rôle in the proof of Szemerédi's theorem the regularity lemma has been widely and successfully applied in many other contexts and we refer to [66, 73] for surveys about the regularity lemma and its applications. At the same time the lemma itself has been the object of many extensions and variations which led to insights into many other aspects of the notion of quasi-randomness.

Extensions and variations In [9] Alon, Duke, Lefmann, Rödl and Yuster addressed the algorithmic aspect of Szemerédi's regularity lemma. In particular, they showed that a partition as guaranteed in Theorem 3 can be found in polynomial time.

Furthermore, Kohayakawa [61] and Rödl (unpublished) have extended Szemerédi's regularity lemma to sparse graphs without too dense induced subgraphs. Their notion of regularity is obtained from (1.1) by scaling by the global density $p = p(n) = |E|/(|X||Y|)$. More precisely, a pair (X, Y) is (ϵ, p)-regular if

$$|d(A, B) - d(X, Y)| < \epsilon p \tag{1.2}$$

holds for all $A \subset X$ and $B \subset Y$ with $|A| > \epsilon|X|$ and $|B| \geq \epsilon|Y|$. This notion of regularity generalises the notion given by Szemerédi and provides control even for sparse graphs. The drawback, unfortunately, is that regular partitions for this notion of regularity do not necessarily exist, the obstacle being locally "too dense" subgraphs. However, all graphs without this obstacle allow a regular partition analogously to Theorem 3.

The extensions of Szemerédi's regularity lemma to hypergraphs are manifold, see e.g. [19, 29, 47, 90, 107] with many breakthroughs in the recent years. Introducing them, however, is beyond the scope of this introduction. Moreover, for our purposes, we will only need to the so-called weak regularity lemma, a straightforward extension of Szemerédi's regularity lemma to uniform hypergraphs which will be introduced in Section 2.2.1.

Another line of research, known under the name quasi-random graphs (or pseudo-random graphs), has substantially enriched the knowledge about the field. It was initiated by A. Thomason [109, 110] and deeply impacted by the work of Chung, Graham, and Wilson [22]. Technically originated in the non-partite form of the notion of regularity, the theory of quasi-random graphs provides many alternative characterisations of the notion of quasi-randomness and we refer to Section 5.1 for a short introduction into the history of the theory of quasi-random graphs and its relation to the algorithmic regularity lemma.

1.2. Summary of the main results

With Szemerédi's regularity lemma as the starting point, two roads diverge in this thesis reflected in the separation of this scripture into the respective parts. On the one hand, the concept of regularity will be applied in the context of extremal hypergraph theory and on the other hand we will investigate the algorithmic and other aspects of the notion of regularity itself.

The results we aim to present in Part I are taken from two papers: the first one is a collaboration with Mathias Schacht [53] and second is a joint work with Yury Person and Mathias Schacht [54]. Part II contains results obtained in joint work with Noga Alon, Amin Coja-Oghlan, Mihyun Kang, Vojtěch Rödl and Mathias Schacht [10].

In this section we want to briefly introduce these results and say a few words about the organisation of the thesis. To put the results into a wider historical context but keeping this introduction short at the same time we will omit much background information and motivations and defer these discussions to the introductions of the respective parts where we think they better fit.

1.2.1. Dirac type theorems for uniform hypergraphs

The problem dealt with in Part I has its root in the realm of the spanning subgraph containment problem which in turn is at the heart of extremal graph theory. The question of interest concerns minimum degree conditions of n-vertex graphs G which force G to contain a *spanning* substructure, i.e. a subgraph isomorphic to an n-vertex graph F. A classical example of such a result is Dirac's theorem on Hamilton cycles, stating that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ exhibits a spanning (so-called Hamilton) cycle, a cycle on n vertices.

In the context of the spanning subgraph problem Szemerédi's regularity lemma has been extensively applied, leading to many deep results so that it is merely an exaggeration to say that its application has boosted this field into another level. We refer to Section 2.1 for a short introduction into the subject and the rôle of the regularity lemma therein.

The main purpose of Part I is to study the spanning sub*hyper*graph containment problem, a straightforward extension of the above question to uniform hypergraphs. In contrast to graphs the situation for hypergraphs is rather unsatisfactory, with very few bounds for very few structures known.

Throughout the thesis we denote by $\mathcal{H} = \mathcal{H}^k$ a k-uniform hypergraph, i.e., a pair $\mathcal{H} = (V, E)$ with the vertex set $V = V(\mathcal{H})$ and the edge set $E = E(\mathcal{H}) \subseteq \binom{V}{k}$, where $\binom{V}{k}$ denotes the family of all k-element subsets of the set V. Given a k-uniform hypergraph $\mathcal{H} = (V, E)$ and a set $R \in \binom{V}{r}$ let $\deg(R)$ denote the number of edges of $\mathcal{H}$ containing the set R and let $\delta_r(\mathcal{H})$ be the minimum r-degree of $\mathcal{H}$, i.e., the minimum of $\deg(R)$ over all r-element sets $R \subseteq V$.

Loose Hamilton cycles in uniform hypergraphs In Chapter 3 we present a Dirac type result for Hamilton cycles in uniform hypergraphs. For $1 \leq \ell < k$ a Hamilton ℓ-cycle in a k-uniform, n-vertex hypergraph is an ordering of the vertices and an ordered subset of the edges such that each such edge contains k consecutive (modulo n) vertices and two consecutive edges intersect in precisely ℓ vertices. Note that for the k-uniform hypergraph to contain an ℓ-cycle it is necessary that n is a multiple of $k-\ell$ which we indicate by $n \in (k-\ell)\mathbb{N}$.

We study sufficient minimum $(k-1)$-degree conditions for the appearance of Hamilton ℓ-cycles in k-uniform hypergraphs. This research was initiated by Katona and Kierstead [58]. These authors considered the case $\ell = k-1$ and such ℓ-cycles are sometimes called *tight* cycles. They showed that $\delta_{k-1}(\mathcal{H}) \geq (1-\frac{1}{2k})|V(\mathcal{H})| - k + 4 - \frac{5}{2k}$ implies the existence of a tight Hamilton path in a k-uniform hypergraph $\mathcal{H}$. The same authors suggested that, in fact, $\delta_{k-1}(\mathcal{H}) \geq (n-k+2)/2$ should suffice and they gave a matching lower bound construction. Recently, Rödl, Ruciński, and Szemerédi [93, 96] answered their question approximately and showed that k-uniform hypergraphs $\mathcal{H}$ on $|V| = n$ vertices with $\delta_{k-1}(\mathcal{H}) \geq (1/2 + o(1))n$ contain Hamilton $(k-1)$-cycles.

Contrasting their result, we provide essentially sharp bounds for *loose* Hamilton cycles, i.e. ℓ-cycles for $\ell < k/2$. The first result concerning (loose) Hamilton 1-cycles for 3-uniform hypergraphs is due to Kühn and Osthus [79] who showed that 3-uniform hypergraphs $\mathcal{H}$ satisfying $\delta_2(\mathcal{H}) \geq (1/4 + o(1))n$ contains a Hamilton 1-cycle. They also showed that this result is best possible up to the error term $o(1)$ and conjectured that $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-1)} + o(1))n$ should force Hamilton 1-cycles in k-uniform hypergraphs. Using the weak regularity lemma for hypergraphs and the "absorption technique" of Rödl, Ruciński, and Szemerédi introduced in [93], we verify this conjecture and prove, more generally, the analogous result for ℓ-cycles with $\ell < k/2$.

Theorem 4. *For all integers $k \geq 3$, all $1 \leq \ell < k/2$ and all $\gamma > 0$ there exists an n_0 such that every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geq n_0$ vertices with $n \in (k-\ell)\mathbb{N}$ and $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-\ell)} + \gamma)n$ contains a Hamilton ℓ-cycle.*

This theorem is best possible up to the error term γ (see Proposition 27) and answers a generalised conjecture of Kühn and Osthus (who conjectured the corresponding bound for $\ell = 1$ and verified it for $k = 3$).

For the case $\ell = 1$ this bound was proven independently by Keevash, Kühn, Mycroft and Osthus [60]. However, their approach uses the Blow-up lemma for hypergraphs [59] and is substantially different from ours. Moreover, very recently Kühn, Mycroft, and Osthus [83] extended our result and show that, indeed, if k is not a multiple of $(k-\ell)$ and $n \in (k-\ell)\mathbb{N}$, then

$$\delta_{k-1}(\mathcal{H}) \geq \left(\frac{1}{\lceil k/(k-\ell) \rceil (k-\ell)} + o(1)\right) n$$

is sufficient to guarantee a Hamilton ℓ-cycle.

Perfect and nearly perfect matchings in uniform hypergraphs In Chapter 4 we study sufficient r-degree conditions for the appearance of perfect and nearly perfect matchings in uniform hypergraphs. Perfect matchings are probably the first non-trivial spanning structure but despite the vivid interests in the recent years the problem is still far from being fully resolved which may be seen as a hint to the difficulty of the spanning subhypergraph containment in general.

For $(k-1)$-minimum degree, however, the situation is very well understood due to the works of various researchers [80, 94, 97]. In particular, Rödl, Ruciński, and Szemerédi showed an exact bound for $(k-1)$-minimum degree which enforces perfect matchings in k-uniform hypergraphs [97]. For such an $\mathcal{H}$ on $n \in k\mathbb{N}$ vertices it is given by $\delta_{k-1}(\mathcal{H}) \geq n/2 - k + c_{k,n}$ where $c_{k,n} \in \{3/2, 2, 5/2, 3\}$ depends on the parity of k and n. This is the only exact result for a spanning subhypergraph containment problem we know of.

On the other hand, the same problem is still wide open for vertex minimum degree (i.e. for $r = 1$) and as a main result we will take the first step to answering this problem by providing an asymptotically tight bound on the minimum vertex degree which ensures a perfect matchings in 3-uniform hypergraphs.

Theorem 5. *For all $\gamma > 0$ there exists an n_0 such that for all $n > n_0$, $n \in 3\mathbb{N}$ the following holds. Suppose $\mathcal{H}$ is a 3-uniform hypergraph on n vertices satisfying $\delta_1(\mathcal{H}) \geq (5/9 + \gamma)\binom{n}{2}$. Then $\mathcal{H}$ contains a perfect matching.*

To our best knowledge, this is the only (asymptotically sharp) result for a spanning subhypergraph containment problem (for $k \geq 3$) which involves vertex minimum degree and it answers a question posed by Kühn and Osthus in [80].

Addressing the case of arbitrary $r < k$ we provide general upper bounds on the minimum r-degree which ensure the existence of perfect and nearly perfect matchings in k-uniform hypergraphs. For $r = k - 1$ a phenomenon, noted e.g. in [80, 94], is that nearly perfect matchings, i.e. matchings covering all but a constant number of vertices (depending on k only), already appear at minimum degree n/k rather than roughly $n/2$. Generalising this result, we show the following upper bound for the existence of nearly perfect matchings in k-uniform hypergraphs.

Theorem 6. *For all integers $k > r > 0$ there is an n_0 such that for all $n > n_0$ the following holds. Suppose $\mathcal{H}$ is a k-uniform hypergraph on $n > n_0$ vertices, $n \in k\mathbb{N}$ with minimum r-degree*

$$\delta_r(\mathcal{H}) \geq \frac{k-r}{k}\binom{n}{k-r} + k^{k+1}(\ln n)^{1/2} n^{k-r-1/2},$$

then $\mathcal{H}$ contains a matching covering all but $(r-1)k$ vertices. In particular, for $r = 1$ the matching is perfect.

Theorem 6 will be derived from a k-partite analogue result (see Theorem 47) and together with the absorption technique, developed by Rödl, Ruciński, and Szemerédi, we obtain the following theorem about the existence of perfect matchings in k-uniform hypergraphs.

Theorem 7. *For all $\gamma > 0$ and all integers $k > r > 0$ there is an n_0 such that for all $n > n_0$, $n \in k\mathbb{N}$ the following holds. Suppose $\mathcal{H}$ is a k-uniform hypergraph on $n > n_0$ vertices with minimum degree*

$$\delta_r(\mathcal{H}) \geq \left(\max\left\{\frac{1}{2}, \frac{k-r}{k}\right\} + \gamma\right)\binom{n}{k-r}$$

then $\mathcal{H}$ contains a perfect matching.

For $r \geq k/2$, the maximum is $n/2$, and this bound is best possible up to the error term γ, which was already shown by Pikhurko [89]. For small r (compared to k) there is a gap between the bound in Theorem 7 and currently best lower bounds and we believe that the bound in Theorem 7 can be improved. In particular, for $k = 3$ and $r = 1$ the maximum is $2/3$ but from Theorem 5 we know that $5/9$ is sufficient.

We note that the proofs of the results concerning perfect matchings do not involve the regularity lemma. However, since perfect matchings are very elementary structures, they have the potential to be applied in combination with the regularity lemma, as shown, e.g., by the proofs of Rödl, Ruciński and Szemerédi in [93, 96] both applying results on matchings as part.

1.2.2. Algorithmic regularity lemma and quasi-random graphs

As mentioned before Alon, Duke, Lefmann, Rödl and Yuster [9] provided an algorithmic version of Szemerédi's regularity lemma and Kohayakawa [61] and Rödl (unpublished) extended the regularity lemma to sparse graphs with nearly regular degree distribution for which there is no algorithmic version known so far. Moreover, many other aspects of notion of quasi-randomness have been extensively studied in the field of quasi-random graphs. In particular, there are many characterisations of the notion of quasi-randomness known.

In Part II we will pursue these lines of research and introduce a new concept of regularity which allows us to deal with (sparse) graphs with general degree distribution, including but not limited to the ubiquitous power-law degree distributions (cf. [2]). Two aspects concerning this new notion of regularity will be investigated. On the one hand, a polynomial time algorithm for finding a regular partition (with respect to this new notion of regularity) will be presented which entails a generalisation of the two results mentioned above [9, 61]. On the other hand, we will provide a characterisation of the notion of quasi-randomness arising from this new concept of regularity. This characterisation will be given in terms of eigenvalue separation of the normalised Laplacian matrix.

An introduction to the algorithmic versions of the regularity lemma and to quasi-random graphs can be found in Section 5.1. We now describe the results in details.

Algorithmic regularity lemma and MAX-CUT approximation The regular partitions which we consider in Chapter 6 take into account a given "ambient" weight distribution $\boldsymbol{D} = (D_v)_{v \in V}$, which is an arbitrary sequence of rationals between 1 and $n = |V|$. Let

$G = (V, E)$ be a graph and for a subset $U \subset V$ let $D(U) = \sum_{u \in U} D_u$. Further, for sets $X, Y \subset V$ we say the pair (X, Y) is **$(\epsilon, \boldsymbol{D})$-regular** if

$$\left| e(X', Y') - e(X, Y) \frac{D(X')D(Y')}{D(X)D(Y)} \right| \leq \epsilon \cdot \frac{D(X)D(Y)}{D(V)} \tag{1.3}$$

is satisfied for all $X' \subset X$, $Y' \subset Y$ with $D(X') \geq \epsilon D(X)$, $D(Y') \geq \epsilon D(Y)$. Roughly speaking, (1.3) states that the number of edges between every big subpair (X', Y') is close to what we expect according to their vertex weights, i.e. the bipartite graph spanned by X and Y is "quasi-random" with respect to the vertex weights $\boldsymbol{D}$.

In the present notation, we obtain Szemerédi's notion of regularity via $D_v = n$ for all $v \in V$. Moreover, with $D_v = \bar{d}$ for a number $\bar{d} = \bar{d}(n) = o(n)$ we obtain the notion of regularity for sparse graphs due to Kohayakawa [61] and Rödl (unpublished). But with respect to such "sparse" weight distributions regular partitions do not necessarily exist, the basic obstacle being the presence of large "dense spots" (X, Y), where $e(X, Y)$ is far bigger than the term $D(X)D(Y)$ suggests. To rule these out, we consider the following notion.

$(C, \eta, \boldsymbol{D})$-boundedness. Let $C \geq 1$ and $\eta > 0$. We call graph G **$(C, \eta, \boldsymbol{D})$-bounded** if $e(X, Y) \frac{D(V)}{D(X)D(Y)} \leq C$ holds for all subsets $X, Y \subset V$ with $D(X), D(Y) \geq \eta D(V)$.

We note that for the sequence $D_v = n$ every graph is $(C, \eta, \boldsymbol{D})$ bounded for all $C \geq 1$ and $\eta > 0$ and for the sequence $D_v = \bar{d}$ with $\bar{d} = \bar{d}(n) = o(n)$ this notion of boundedness coincides with the one of Kohayakawa and Rödl.

Based on the notion of regularity given in (1.3) we provide an *algorithmic regularity lemma* for graphs with general degree distributions, which is in particular an algorithmic version for Szemerédi's regularity lemma and the regularity lemma for sparse graphs due to Kohayakawa and Rödl. The new concept, however, allows graphs with highly irregular degree distributions.

Let $\langle \boldsymbol{D} \rangle$ signify the **encoding length** of a weight distribution $\boldsymbol{D} = (D_v)_{v \in V}$, i.e., the number of bits that are needed to write down the rationals $(D_v)_{v \in V}$. Observe that $\langle \boldsymbol{D} \rangle \geq n$.

Theorem 8. *For any two numbers $C \geq 1$ and $\epsilon > 0$ there exist $\eta > 0$ and $n_0 > 0$ such that for all $n \geq n_0$ and every sequence of rationals $\boldsymbol{D} = (D_v)_{v \in V}$ with $|V| = n$ and $1 \leq D_v \leq n$ for all $v \in V$ the following holds. If $G = (V, E)$ is a $(C, \eta, \boldsymbol{D})$-bounded graph and $D(V) \geq \eta^{-1} n$, then there is a partition $\mathcal{P} = \{V_i \colon 0 \leq i \leq t\}$ of V that satisfies the following two properties:*

REG1. *(a) $\eta D(V) \leq D(V_i) \leq \epsilon D(V)$ for all $i = 1, \ldots, t$,*

(b) $D(V_0) \leq \epsilon D(V)$, and

(c) $|D(V_i) - D(V_j)| < \max_{v \in V} D_v$ for all $1 \leq i < j \leq t$.

REG2. *Let $\mathcal{L}$ be the set of all pairs (i, j), $1 \leq i < j \leq t$ such that (V_i, V_j) is not $(\epsilon, \boldsymbol{D})$-regular. Then*

$$\sum_{(i,j) \in \mathcal{L}} D(V_i)D(V_j) \leq \epsilon D(V)^2.$$

Furthermore, for fixed C and ϵ the partition $\mathcal{P}$ can be computed in polynomial time. More precisely, there exist a function f and a polynomial Π such that the partition $\mathcal{P}$ can be computed in time $f(C, \epsilon) \cdot \Pi(\langle \boldsymbol{D} \rangle)$.

Condition **REG1** states that all of the classes $V_1, \ldots, V_t$ have approximately the same, non-negligible weight, while the "exceptional" class V_0 has a "small" weight. Also note that due to **REG1(a)** the number of classes t of the partition $\mathcal{P}$ is bounded by $1/\eta$, which only depends on C and ϵ, but not on G, $\boldsymbol{D}$, or n. Moreover, **REG2** requires that the total weight of the irregular pairs (V_i, V_j) is small relative to the total weight. Thus, a partition $\mathcal{P}$ that satisfies **REG1** and **REG2** approximates G by a bounded number of bipartite quasi-random graphs.

We illustrate the use of Theorem 8 with the example of the MAX-CUT problem. While approximating MAX-CUT within a ratio better than $\frac{16}{17}$ is NP-hard on general graphs [55, 111], the following theorem provides a polynomial time approximation scheme for $(C, \eta, \boldsymbol{D})$-bounded graphs.

Theorem 9. *For any $\delta > 0$ and $C \geq 1$ there exist two numbers $\eta > 0$, $n_0 > 0$ and a polynomial time algorithm* `ApxMaxCut` *such that for all $n \geq n_0$ and every sequence of rationals $\boldsymbol{D} = (D_v)_{v \in V}$ with $|V| = n$ and $1 \leq D_v \leq n$ for all $v \in V$ the following is true. If $G = (V, E)$ is a $(C, \eta, \boldsymbol{D})$-bounded graph and $D(V) > \eta^{-1} n$, then* `ApxMaxCut` *outputs a cut of G that approximates the maximum cut up to an additive error of $\delta |D(V)|$.*

For all $v \in V$, by taking D_v to be the degree of v and noting that the maximum cut of a graph G is at least $e(G)/2$ we immediately derive from Theorem 9 that the problem MAX-CUT can be approximated within a multiplicative factor $(1 + o(1))$ in polynomial time for the class of $(C, \eta, \boldsymbol{D})$-bounded graphs.

Quasi-randomness: Low discrepancy and eigenvalue separation As mentioned before, the notion of regularity (1.1) due to Szemerédi has a natural non-partite analogue which was subject of extensive study in the theory of quasi-random graphs. This notion, sometimes called low discrepancy, is given by

$$\left| e(S) - p \binom{|S|}{2} \right| < \epsilon n^2 \quad \text{for all } S \subset V \tag{1.4}$$

where $p = |E| / \binom{n}{2}$ denotes the density of $G = (V, E)$. The analogy to Szemerédi's concept of regularity is evident since (1.4) essentially asserts that the density of G and its induced subgraphs is small.

Since the work of Chung, Graham, and Wilson [22] there are many characterisations of this "dense" notion of low discrepancy known and we refer to Section 5.1 for a short introduction into the theory of quasi-random graphs.

Chapter 7 pursues this line of research, aiming at a characterisation of the notion of $(\epsilon, \boldsymbol{D})$-regularity as given in (1.3). For a set $S \subset V$ we define $\mathrm{vol}(S) = \sum_{v \in S} d_v$. With $X = Y = V$, $X' = Y' = S$ and $D(S) = \mathrm{vol}(S)$ the following property, introduced by Chung and Graham in [21], is immediately derived from the definition of $(\epsilon, \boldsymbol{D})$-regularity (1.3).

Disc(ϵ): We say that G **has discrepancy** ϵ ("G has Disc(ϵ)" for short) if

$$\forall S \subset V : \left| e(S) - \frac{\mathrm{vol}(S)^2}{2\mathrm{vol}(V)} \right| < \epsilon \cdot \mathrm{vol}(V). \tag{1.5}$$

To explain (1.5), let $\boldsymbol{d} = (d_v)_{v \in V}$, and let $G(\boldsymbol{d})$ signify a random graph with expected degree distribution $\boldsymbol{d}$; that is, any two vertices v, w are adjacent with probability $p_{vw} = d_v d_w / \mathrm{vol}(V)$ independently. Then in $G(\boldsymbol{d})$ the *expected* number of edges inside of $S \subset V$ equals $\frac{1}{2} \sum_{(v,w) \in S^2} p_{vw} = \frac{1}{2} \mathrm{vol}(S)^2 / \mathrm{vol}(V)$. Consequently, (1.5) just says that for *any* set S the actual number of edges inside of S must not deviate from what we expect in $G(\boldsymbol{d})$ by more than an ϵ-fraction of the total volume.

Compared to the dense notion of low discrepancy mentioned in (1.4) much less is known for Disc(ϵ). In particular, in the dense case there is a characterisation of low discrepancy in terms of the eigenvalues (of the adjacency matrix) and providing such a characterisation for Disc(ϵ) has been an open problem in the area of sparse quasi-random graphs and quasi-random graphs with general degree distribution since the works of Chung and Graham [17, 21].

In Chapter 7 we present such a characterisation in terms of the eigenvalues of the **normalised Laplacian matrix** of G. This matrix $L(G) = (\ell_{vw})_{v,w \in V}$ is given by

$$\ell_{vw} = \begin{cases} 1 & \text{if } v = w \text{ and } d_v \geq 1, \\ -(d_v d_w)^{-\frac{1}{2}} & \text{if } v, w \text{ are adjacent}, \\ 0 & \text{otherwise}; \end{cases}$$

Due to the normalisation by the geometric mean $\sqrt{d_v d_w}$ of the vertex degrees, $L(G)$ turns out to be appropriate for representing graphs with general degree distributions. Moreover, $L(G)$ is well known to be positive semidefinite, and the multiplicity of the eigenvalue 0 equals the number of connected components of G (we refer to Section 5.2.1 and Section 5.2.2 for a short introduction). Consider the following property for the Laplacian.

Eig(δ): Letting $0 = \lambda_1 [L(G)] \leq \cdots \leq \lambda_{|V|} [L(G)]$ denote the eigenvalues of $L(G)$, we say that G **has δ-eigenvalue separation** ("G has Eig(δ)") if

$$1 - \delta \leq \lambda_2 [L(G)] \leq \lambda_{|V|} [L(G)] \leq 1 + \delta. \tag{1.6}$$

As the eigenvalues of $L(G)$ can be computed in polynomial time (within arbitrary numerical precision), we can essentially check efficiently whether G has Eig(δ) or not.

It is not difficult to see that Eig(δ) provides a *sufficient* condition for Disc(ϵ). That is, for any $\epsilon > 0$ there is a $\delta > 0$ such that any graph G that has Eig(δ) also has Disc(ϵ). However, while the converse implication is true if G is dense (i.e., $\mathrm{vol}(V) = \Omega(|V|^2)$), it is false for sparse graphs. Concerning this problem, we basically observe that the reason why Disc(ϵ) does in general not imply Eig(δ) is the existence of a small set of "exceptional" vertices. With this in mind we refine the definition of Eig as follows.

ess-Eig(δ): We say that the graph G **has essential δ-eigenvalue separation** ("G has ess-Eig(δ)") if there is a set $W \subset V$ of volume $\mathrm{vol}(W) \geq (1-\delta)\mathrm{vol}(V)$ such that the following is true. Let $L(G)_W = (\ell_{vw})_{v,w\in W}$ denote the minor of $L(G)$ induced on $W \times W$, and let $\lambda_1 [L(G)_W] \leq \cdots \leq \lambda_{|W|} [L(G)_W]$ signify its eigenvalues. Then we require that

$$1 - \delta < \lambda_2 [L(G)_W] \leq \lambda_{|W|} [L(G)_W] < 1 + \delta. \tag{1.7}$$

Proving the following theorem we establish the equivalence mentioned above.

Theorem 10. *There is a constant $\gamma > 0$ such that the following is true for all graphs $G = (V, E)$ and all $\epsilon > 0$.*

1. *If G has* ess-Eig(ϵ)*, then G satisfies* Disc($20\sqrt{\epsilon}$).
2. *If G has* Disc($\gamma\epsilon^2$)*, then G satisfies* ess-Eig(ϵ).

The main contribution is the second implication. Its proof is based on Grothendieck's inequality and the duality theorem for semidefinite programs. In effect, the proof actually provides us with an efficient algorithm that computes a set W as in the definition of ess-Eig(ϵ).

Moreover, the techniques presented in the proof can be adapted easily to obtain a similar result as Theorem 10 with respect to the concepts of discrepancy and eigenvalue separation for the adjacency matrix in the case of sparse graphs [17]. More precisely, let $G = (V, E)$ be a graph on n vertices, let $p = 2|E|n^{-2}$ be the edge density of G, and let $\gamma > 0$ denote a small enough constant. If for any subset $X \subset V$ we have $|2e(X) - |X|^2 p| < \gamma\epsilon^2 n^2 p$, then there exists a set $W \subset V$ of size $|W| \geq (1-\epsilon)n$ such that the following is true. Letting $A = A(G)$ signify the adjacency matrix of G, we have $\max\{-\lambda_1 [A_W], \lambda_{|W|-1} [A_W]\} \leq \epsilon np$. That is, all eigenvalues of the minor A_W except for the largest are at most ϵnp in absolute value.

1.3. Organisation

As mentioned before, the thesis is divided into two parts with short introductions into the history of the respective problems. More precisely, Section 2.1 will shortly survey few major results on spanning subgraph containment with an emphasis on the rôle of what is often called the regularity method in this discourse. Section 5.1 gives a brief overview on the history of quasi-random graphs and its connections to the algorithmic versions of the regularity lemma.

The proofs of the results described above are contained in Chapter 3 and Chapter 4 in Part I, and Chapter 6 and Chapter 7 in for Part II. So as to make this scripture more self-contained we will provide the relevant tools for each part separately. In particular, Section 2.2 contains a short introduction into the weak regularity lemma for uniform hypergraphs as well as some basic concentration results for random variables and results in extremal graph theory. Section 5.2 contains a short introduction into positive semidefinite matrices, the Laplacian matrix, semidefinite programming and Grothendieck's inequality.

Part I.

Dirac Type Theorems for Uniform Hypergraphs

2. Prerequisites

The main purpose of Part I is to present the results on spanning subhypergraph containment as introduced in Section 1.2.1. In Chapter 3 we show such a result for loose Hamilton cycles. It was obtained in collaboration with Mathias Schacht [53]. Chapter 4 contains results on perfect and nearly perfect matchings, a joint work with Yury Person and Mathias Schacht [54]. Both works deal with a typical question in extremal graph theory whose history we will therefore briefly introduce in the following in order to locate the results in their wider context. In particular, we will discuss the corresponding problem for graphs, i.e. the spanning subgraph containment problem, and we will put an emphasis on the rôle of the regularity method in this discourse.

2.1. Historical background

Extremal graph theory is among the most popular and best studied fields of combinatorics. A fundamental question in this area concerns about degree conditions of graphs G under which they must exhibit certain substructures, e.g. subgraphs isomorphic to a given graph F.

The birth of extremal graph theory If F is, say, the complete graph K_k on k vertices, then the following result by Turán from 1941 [113] gives a sharp bound on the average degree of G which ensures a copy of $F = K_k$ in G.

Theorem 11 (Turán)**.** *Given two positive integers k and n and let $T_{n,k-1}$ denote the $(k-1)$ partite graph on n vertices with the partition classes being as equal as possible. Then $T_{n,k-1}$ maximises the number of edges among all graphs on n vertices not containing a copy of K_k. (Thus, the average degree of any K_k-free graph is at most $(k-2)n/(k-1)$.)*

This theorem is considered the date of birth of Extremal graph theory, although the first results in this flavour [26, 86] including a special case of Turán's theorem itself [86] have already been proven much earlier.

Local subgraph containment Moving from K_k to arbitrary F of fixed order Erdős and Stone showed that $\chi(F)$, the chromatic number of F, is the parameter to look at. In [27] they proved the following.

Theorem 12 (Erdős & Stone)**.** *Let F be a fixed graph with chromatic number k and let G be an n-vertex graph which does not contain a copy of F. Then the average degree $\bar{d}(G)$ of G satisfies*

$$\bar{d}(G) \leq \left(\frac{k-2}{k-1} + o(1)\right) n.$$

This theorem, which is often referred to as the "fundamental theorem of extremal graph theory" [12] resolves the subgraph containment problem to a major extent as long as F is of small order (compared to $|V(G)| = n$, the order of G).

Spanning subgraph containment The focus of our consideration, however, is the other important case, when the order of F with the order of G, in particular when $|V(F)| = |V(G)|$. Obviously, the average degree of G is not the right parameter to look at in this context, since very dense graphs G may still contain isolated vertices. Appealing to the minimum degree of G, therefore, seems to be natural and turned out to be fruitful at the same time.

Results in this context are copious and surveying them all is far beyond our narrow scope. Therefore, we will restrict ourselves to a very elementary introduction, referring the reader to the surveys [114] and [81] for a thorough treatment and further details.

We start with Dirac's theorem on Hamilton cycles [25], one of the first and concurrently most well known result in the context of spanning subgraph containment.

Theorem 13 (Dirac '52)**.** *Every graph G with $n \geq 3$ vertices and minimum degree at least $n/2$ contains a Hamilton cycle.*

It is easily seen that $n/2$ is best possible. For odd n the complete bipartite graph with the sizes of the partition classes being $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ contains a Hamilton path but does not contain a Hamilton cycle whereas for even n the graph consisting of two cliques of size $n/2$ is not even connected.

In succession many classical results appeared, one of them being the following on triangle factor due to Corrádi and Hajnal [24].

Theorem 14 (Corrádi & Hajnal '63)**.** *Every n-vertex graph G with minimum degree $\delta(G) \geq \frac{2}{3}n$ contains $\lfloor n/3 \rfloor$ vertex disjoint copies of K_3.*

Again, Theorem 14 is best possible as seen from the complete tripartite graph with the sizes of the partition classes being $\lfloor n/3 \rfloor - 1$, $\lceil n/3 \rceil + 1$, and $n - \lfloor n/3 \rfloor - \lceil n/3 \rceil$. Later this result was generalised to K_k-factor by Hajnal and Szemerédi [51].

Theorem 15 (Hajnal & Szemerédi '70)**.** *Every large n-vertex graph G with minimum degree $\delta(G) \geq \frac{k-1}{k}n$ contains $\lfloor n/k \rfloor$ vertex disjoint copies of K_k.*

This line of research continued and culminated in the proof of the Pósa-Seymour conjecture [99] due to Komlós, Sárközy, and Szemerédi [67, 69, 70].

Theorem 16 (Komlós, Sárközy & Szemerédi '96)**.** *Every n-vertex graph G with minimum degree $\delta(G) \geq \frac{k-1}{k}n$ contains a copy of the $(k-1)$st-power of a Hamilton cycle.*

Here, the kth-power of a graph F is obtained by connecting distinct vertices of distance at most k in F. By definition, the first power of a Hamilton cycle is the Hamilton cycle itself and the kth-power of a Hamilton cycle contains $\lfloor n/k \rfloor$ vertex disjoint K_k. Thus, this result generalises all the result mentioned so far, i.e. Theorem 13, Theorem 14, and

Theorem 15. Moreover, it is best possible as seen from a straightforward extension of the example given for Theorem 14.

Many other results in this flavour are known, including e.g. results on spanning trees, spanning planar graphs, F-factors [8, 72, 65, 82, 84, 78] and we close the historical introduction with the following sharp result [15] which was conjectured by Bollobás and Komlós [64] (see [1] for a proof of sharpness). In the following, a graph is said to have bandwidth at most b, if there exists a labelling of the vertices by numbers $1, ..., n$, such that every edge $\{i, j\}$ of the graph satisfies $|i - j| \le b$.

Theorem 17 (Böttcher, Schacht & Taraz '09)**.** *For all k, $\Delta \in \mathbb{N}$ and $\gamma > 0$ there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ the following holds. If H is an k-chromatic graph on n vertices with maximum degree $\Delta(H) \le \Delta$ and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \ge (\frac{k-1}{k} + \gamma)n$, then G contains a copy of H.*

The rôle of the regularity method The regularity method, most notably Szeremédi's regularity lemma (Theorem 3) and the blow-up lemma, has played a crucial rôle in the discourse. The proofs of many results mentioned above heavily depend on both theorems and the corresponding proofs without application of the regularity method seems to be out of reach for many (see [85] for an exception). They all follow a general pattern with some problem specific adaptions.

To prove an approximated result (i.e. not a sharp one but allowing an additional error γn for any small constant $\gamma > 0$) the regularity method is often applied as follows. Let the graph G be given in which we want to embed F and suppose G has minimum degree, say, $\delta(G) \ge (x + \gamma)n$. First, we prepare F by chopping it into a constant number of small subgraphs. Then we apply the regularity lemma to G to obtain a regular partition $V_1, \dots, V_t$. We define the cluster graph of G with the vertices being the partition classes V_i, $1 \le i \le t$, and the edges being those pairs which are regular and dense. Thus, this is a t-vertex graph and a nice feature of the regularity lemma is that the cluster graph of G almost inherit the minimum degree of G, i.e. the cluster graph will have minimum degree at least $(x + \gamma/2)t$. Thus, one can apply one of the known results on spanning subgraph containment to obtain a spanning structure on the reduced graph, e.g., to obtain a K_k-factor (i.e. for $x \ge \frac{k-1}{k}$). At this point, we assign the prepared pieces of F to the k-cliques found in cluster graph and are left with three problems. First, there may be leftover vertices which we have to care about, second, we may have to connect the the k-cliques and thirdly, we have to embed the assigned pieces into the "regular k-cliques". The first two tasks highly depend on the problem itself and is handled in very different ways. For the last task, however, we can appeal to the blow-up lemma which was developed by Komlós, Sarközy, and Szemerédi [68, 71]. Roughly speaking, the blow-up lemma states that regular pairs with additional minimum degree condition behave like complete bipartite graphs with respect to embedding spanning graphs with bounded degree.

Lastly, for the proofs of exact bounds for spanning subgraph containment problems one distinguishes two cases which are handled separately. Either the given graph G

is close to the extremal counter example or it is “far away” from it. In the first case the embedding of F is constructed by hand and in the second case one can follow the strategy given above.

The spanning subhypergraph containment problem As seen from above the theory of spanning subgraph containment is fairly well developed for graphs. Thus, it is natural to extend the corresponding question to k-uniform hypergraphs, $k \geq 3$ and this problem has attracted the interest of many researchers in the recent years.

One reason for the interest may lie in the development of the tools for tackling those problems, most notably the regularity lemmas for hypergraphs which have reached maturity due to works of various researchers (see e.g. [47, 90, 107]) and which are much better understood than one decade ago. Moreover, a corresponding blow-up lemma was established for uniform hypergraphs [59].

At the same time the problems seem to be much more challenging and we note that even Turán type problems for hypergraphs, i.e. the subhypergraph containment problems for constant size, is notoriously difficult and is not even fully resolved for $K_4^{(3)}$, the 3-uniform complete hypergraph on 4 vertices (see [35] for a survey). Similarly and in contrast to graphs, the situation for the spanning subhypergraph containment problem ($k \geq 3$) is rather unsatisfactory with very few bounds for very basic structures known.

To the best knowledge of the author, there are no asymptotically sharp bounds for spanning subgraph containment known besides the results on perfect matchings and Hamilton cycles mentioned in Section 1.2. Moreover, there is only one exact result in this field which is on perfect matchings due to Rödl, Ruciński and Szemerédi [97] and in general it seems to be more difficult to prove good bounds for low minimum degree than, say $(k-1)$-degree δ_{k-1}. The only asymptotic sharp result known which involves 1-minimum degree δ_1 is Theorem 5.

2.2. Tools

For completeness we collect the tools needed for the proofs which include very basic results in extremal graph theory, probabilistic inequalities and the weak regularity lemma for uniform hypergraphs. The reader can safely skip this section and come back whenever needed.

2.2.1. The weak hypergraph regularity lemma

In this section we introduce the so-called *weak hypergraph regularity lemma*, a straightforward extension of Szemerédi's regularity lemma [105] for graphs.

Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph and let $A_1, \ldots, A_k$ be mutually disjoint non-empty subsets of V. We define $e_{\mathcal{H}}(A_1, \ldots, A_k)$ to be the number of edges with one vertex in each A_i, $i \in [k]$ and the **density** of $\mathcal{H}$ with respect to $(A_1, \ldots, A_k)$ as

$$d_{\mathcal{H}}(A_1, \ldots, A_k) = \frac{e_{\mathcal{H}}(A_1, \ldots, A_k)}{|A_1| \cdot \ldots \cdot |A_k|} .$$

We say the k-tuple $(V_1, \ldots, V_k)$ of mutually disjoint sets $V_1, \ldots, V_k \subseteq V$ is (ϵ, d)**-regular**, for constants $\epsilon > 0$ and $d \geq 0$, if for all k-tuples of subsets $A_1 \subset V_1, \ldots, A_k \subset V_k$ with $|A_1| \geq \epsilon|V_1|, \ldots, |A_k| \geq \epsilon|V_k|$ the condition

$$|d_{\mathcal{H}}(A_1, \ldots, A_k) - d| \leq \epsilon$$

is satisfied. We say the k-tuple $(V_1, \ldots, V_k)$ is ϵ**-regular** if it is (ϵ, d)-regular for some $d \geq 0$. The following fact is a direct consequence of the definition above.

Fact 18. *For an (ϵ, d)-regular tuple $(V_1, \ldots, V_k)$ we have*

(i) $(V_1, \ldots, V_k)$ *is (ϵ', d)-regular for all $\epsilon' > \epsilon$ and*

(ii) *if for all $i \in [k]$ the set $V_i' \subset V_i$ has size $|V_i'| \geq c|V_i|$, then the tuple $(V_1', \ldots, V_k')$ is $(\epsilon/c, d)$-regular.*

As a straightforward generalisation of the original regularity lemma we obtain the following regularity lemma for graphs (see, e.g., [18, 30, 104]).

Theorem 19 (Weak regularity lemma for hypergraphs)**.** *For integers $k \geq 2$ and $t_0 \geq 1$, and all $\epsilon > 0$, there exist $T_0 = T_0(k, t_0, \epsilon)$ and $n_0 = n_0(k, t_0, \epsilon)$ so that for every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $n \geq n_0$ vertices, there exists a partition $V = V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_t$ such that*

(i) $t_0 \leq t \leq T_0$,

(ii) $|V_1| = |V_2| = \cdots = |V_t|$ *and* $|V_0| \leq \epsilon n$,

(iii) *for all but at most $\epsilon\binom{t}{k}$ sets $\{i_1, \ldots, i_k\} \in \binom{[t]}{k}$, the k-tuple $(V_{i_1}, \ldots, V_{i_k})$ is ϵ-regular.*

A partition as given in Theorem 19 is called an ϵ**-regular partition** of $\mathcal{H}$ (with lower bound t_0 on the number of vertex classes). Further, we need the notion of the cluster graph.

Definition 20. *Given an ϵ-regular partition of $\mathcal{H}$ and $d \geq 0$. We refer to the sets $V_i, i \in [t]$ as **clusters** and define the **cluster hypergraph** $\mathcal{K} = \mathcal{K}(\epsilon, d)$ with vertex set $[t] = \{1, 2, \ldots, t\}$ and $\{i_1, \ldots, i_k\} \in \binom{[t]}{k}$ being an edge if and only if $(V_{i_1}, \ldots, V_{i_k})$ is ϵ-regular and $d(V_{i_1}, \ldots, V_{i_k}) \geq d$.*

The following proposition relates the degree condition of $\mathcal{H}$ and its cluster hypergraph $\mathcal{K}$. It shows that $\mathcal{K}$ "almost inherits" the minimum degree of $\mathcal{H}$.

Proposition 21. *Let $x, \gamma > 0$ and let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph with minimum $(k-1)$-degree*

$$\delta_{k-1}(\mathcal{H}) \geq (x + \gamma)\, n.$$

Further, let $V = V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_t$ be an ϵ-regular partition of $\mathcal{H}$ with $0 < \epsilon < \gamma^2/16$ and $t_0 \geq 8k/\epsilon \geq 3k/\gamma$ and let $\mathcal{K} = \mathcal{K}(\epsilon, \gamma/6)$ be the cluster hypergraph of $\mathcal{H}$. Then the number of $(k-1)$-sets $S = \{i_1, \ldots, i_k\} \in \binom{[t]}{k-1}$ violating

$$\deg_{\mathcal{K}}(S) \geq (x + \gamma/4)\, t$$

is at most $\sqrt{\epsilon}t^{k-1}$.

Proof. Note first that the cluster hypergraph $\mathcal{K}(\epsilon, \gamma/6)$ can be written as the intersection of two hypergraphs $\mathcal{D} = \mathcal{D}(\gamma/6)$ and $\mathcal{R} = \mathcal{R}(\epsilon)$ both defined on the vertex set $[t]$ and

- $\mathcal{D}(\gamma/6)$ consists of all sets $\{i_1, \ldots, i_k\}$ such that $d(V_{i_1}, \ldots, V_{i_k}) \geq \gamma/6$
- $\mathcal{R}(\epsilon)$ consists of all sets $\{i_1, \ldots, i_k\}$ such that $(V_{i_1}, \ldots, V_{i_k})$ is ϵ-regular.

Given an arbitrary set $S \in \binom{[t]}{k-1}$ we first show

$$\deg_{\mathcal{D}}(S) \geq (x + \gamma/2)\, t. \tag{2.1}$$

To this end note that $S = \{i_1, \ldots, i_{k-1}\}$ represents the tuple $(V_{i_1}, \ldots, V_{i_{k-1}})$ with $n/t \geq m := |V_{i_j}| \geq (1-\epsilon)n/t$ for all $j \in [k-1]$. We consider now the number of edges in $\mathcal{H}$ which intersects each V_{i_j} in exactly one vertex. From the condition on $\delta_{k-1}(\mathcal{H})$ this is at least

$$m^{k-1}\left((x+\gamma)\, n - (k-1)m\right) \geq m^{k-1}\left(x + \frac{2\gamma}{3}\right) n \tag{2.2}$$

since $t \geq t_0 \geq 3k/\gamma$.

On the other hand, in case (2.1) does not hold the same number can be bounded from above by

$$\left(x + \frac{\gamma}{2}\right) t \times m^k + t \times \frac{\gamma}{6} m^k$$

with contradiction to (2.2).

Next, observe that there are at most $\epsilon\binom{t}{k} < \epsilon t^k/k$ sets $\{i_1, \ldots, i_k\} \in \binom{[t]}{k}$ such that the corresponding tuples $(V_{i_1}, \ldots, V_{i_k})$ are not ϵ-regular, i.e. $\{i_1, \ldots, i_k\} \notin \mathcal{R}$. Thus, all but at most $\sqrt{\epsilon}t^{k-1}$ sets $S \in \binom{[t]}{k-1}$ satisfy

$$\deg_{\mathcal{R}}(S) \geq (1 - \sqrt{\epsilon})t. \tag{2.3}$$

Since $\mathcal{K} = \mathcal{D} \cap \mathcal{R}$ the proposition follows from (2.1), (2.3) and our choice $\sqrt{\epsilon}t \leq \gamma t/4$. $\square$

2.2.2. Further tools

We will need the following special form of Jensen's inequality.

Theorem 22 (Jensen's inequality)**.** *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a convex function and let* $x_1, x_2, ..., x_n$ *be in its domain, then*

$$f\left(\frac{\sum_{i\in[n]} x_i}{n}\right) \leq \frac{1}{n} \sum_{i\in[n]} f(x_i).$$

Basics from extremal graph theory Extremal graph theory has been introduced in Section 2.1. We will need two other classic results. The first addresses Turán's question for complete bipartite graphs [74].

Theorem 23 (Kővari, Sós & Turán)**.** *Let $s, t \in \mathbb{N}$ and let $K_{s,t}$ denote the complete bipartite graph with s and t being the sizes of the partition classes. If an n-vertex graph G does not contain a copy of $K_{s,t}$ then*

$$e(G) \leq \frac{1}{2}\left((s-1)^{1/t} n^{2-1/t} + tn\right).$$

The theorem of the following type is often called supersaturation. It basically says that graphs G whose size exceed the Turán number contain not only one copy of the forbidden graphs F but many.

Theorem 24. *For all $\epsilon' > 0$ there is a $c = c(\epsilon') > 0$ and $n_0 = n_0(\epsilon')$ such that for all $n \geq n_0$ the following holds. Suppose G is a graph on n vertices which contains at least $(1/2 + \epsilon')\binom{n}{2}$ edges. Then G contains cn^3 triangles.*

Theorem 24 was proven e.g. in [87]. Using Szemerédi's regularity lemma, however, the theorem is almost immediate. To sketch the proof let G be as stated. After applying Szemerédi's regularity lemma, we observe that the t-vertex cluster graph R contains more than $(1 + \epsilon')/2\binom{t}{2}$ edges. Hence, by Turán's theorem (Theorem 11) R contains a triangle which corresponds to three partition classes which pairwise form a regular pair. By applying the Counting Lemma (Theorem 2) we obtain the conclusion of Theorem 24.

Probabilistic tools The following basic probabilistic inequalities can be found in the monographs [7, 57, 13].

Theorem 25 (Markov's inequality)**.** *Let $X \geq 0$ be a random variable and $t > 0$, then*

$$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

Basic concentration results for random variables will also be needed. In particular, we are interested in binomially distributed random variables $X \in \mathrm{Bi(n, p)}$ with parameters n and p, i.e. $X = \sum_{i\in[n]} X_i$ is the sum of independent Bernoulli distributed variables X_i with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = 0] = 1 - p$. Moreover, hypergeometrically distributed random variables with parameter N, n, and m will be used. Given an N element set with m distinguished elements, then the hypergeometric distribution is the distribution of the random variable X defined by drawing n elements from the N elements set and counting the distinguished elements selected via this process. The following result can be found, e.g., in [57].

Theorem 26 (Chernoff's inequality)**.** *Let $X \in \mathrm{Bi(n, p)}$ be a binomially distributed random variable with parameters n and p and let $t \geq 0$. Then*

$$\mathbb{P}[X \geq \mathbb{E}[X] + t] \leq \exp\left(-\frac{t^2}{2\mathbb{E}[X] + t/3}\right) \tag{2.4}$$

$$\mathbb{P}\left[X \leq \mathbb{E}\left[X\right] - t\right] \leq \exp\left(-\frac{t^2}{2\mathbb{E}\left[X\right]}\right). \tag{2.5}$$

Furthermore, (2.4) *and* (2.5) *also holds if X is hypergeometrically distributed with parameter N, n, m, (hence having $\mathbb{E}\left[X\right] = nm/N$).*

3. Loose Hamilton Cycles

In this chapter we want to prove Theorem 4. As mentioned earlier the Theorem 4 is approximately best possible. This is shown by the following straightforward extension of a construction from [79].

Proposition 27. *For every $1 \leq \ell < k/2$ and $n \in 2(k-\ell)\mathbb{N}$ there exists a k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n$ vertices with $\delta_{k-1}(\mathcal{H}) \geq \frac{n}{2(k-\ell)} - 1$, which contains no Hamilton ℓ-cycle.*

Proof. We consider the following k-uniform hypergraph $\mathcal{H} = (V, E)$. Let $A \dot\cup B = V$ be a partition of V with $|A| = \frac{n}{2(k-\ell)} - 1$ and let E be the set of all k-tuples from V with at least one vertex in A. Clearly, we have $\delta_{k-1}(\mathcal{H}) = |A| = \frac{n}{2(k-\ell)} - 1$ and for an arbitrary cycle in $\mathcal{H}$ note the following. Since $\ell < k/2$ every vertex, in particular every vertex from A, is contained in at most 2 edges of this cycle. Moreover, every edge of the cycle must intersect A. Consequently, the cycle contains at most $2|A| < n/(k-\ell)$ edges and, hence, cannot be a Hamilton cycle. $\square$

3.1. Proof of the main theorem

The proof of Theorem 4 follows the approach of Rödl, Ruciński, and Szemerédi from [93] and will be given in Section 3.1.3. This approach is based on three auxiliary lemmas, which we introduce in Section 3.1.2. We start with an outline of the proof.

3.1.1. Outline of the proof

We will build the Hamilton ℓ-cycles by connecting ℓ-paths. An ℓ-path (with distinguished ends) is defined similarly to ℓ-cycles. Formally, a k-uniform hypergraph $\mathcal{P}$ is an **ℓ-path** if there is an ordering $(v_0, \ldots, v_{t-1})$ of its vertices such that every edge consists of k consecutive vertices and two consecutive edges intersect in exactly ℓ vertices. The ordered ℓ-sets $F^{\mathrm{beg}} = (v_0, \ldots, v_{\ell-1})$ and $F^{\mathrm{end}} = (v_{t-\ell}, \ldots, v_{t-1})$ are called the **ends of $\mathcal{P}$**.

Note that this require that $t - \ell$ is a multiple of $k - \ell$. Furthermore, for loose paths (i.e. $\ell < k/2$) the ordering of the ends of an ℓ-path do not matter and we may refer to F^{beg} and F^{end} as sets.

The first lemma, the Absorbing Lemma (Lemma 28), asserts that for $\ell < k/2$ every n-vertex, k-uniform hypergraph $\mathcal{H} = (V, E)$ with $\delta_{k-1}(\mathcal{H}) \geq \epsilon n$ contains a special, so-called *absorbing*, ℓ-path $\mathcal{P}$, which has the following property: For every set $U \subset V \setminus V(\mathcal{P})$ with $|U| \in (k-\ell)\mathbb{N}$ and $|U| \leq \alpha n$ (for some appropriate $0 < \alpha \ll \epsilon$) there exists an ℓ-path $\mathcal{Q}$ with the same ends as $\mathcal{P}$, which covers precisely the vertices $V(\mathcal{P}) \cup U$.

The Absorbing Lemma reduces the problem of finding a Hamilton ℓ-cycle to the simpler problem of finding an almost spanning ℓ-cycle, which contains the absorbing path $\mathcal{P}$ and covers at least $(1-\alpha)n$ of the vertices. We approach this simpler problem as follows. Let $\mathcal{H}'$ be the induced subhypergraph $\mathcal{H}$, which we obtain after removing the vertices of the absorbing path $\mathcal{P}$ guaranteed by the Absorbing Lemma. We remove from $\mathcal{H}'$ a "small" set R of vertices, called *reservoir* (see Lemma 29), which has the property, that every $(k-1)$-tuple of V has "many" neighbours in R. Let $\mathcal{H}''$ be the remaining hypergraph after removing the vertices from R. Note that the property of R allows us to connect every pair $\mathcal{P}_1$ and $\mathcal{P}_2$ of disjoint ℓ-paths in $\mathcal{H}''$ to one ℓ-path, by connecting the end F_1^{end} of $\mathcal{P}_1$ with the beginning F_2^{beg} of $\mathcal{P}_2$ by one edge, where the additional $k-2\ell$ vertices come from R.

The path $\mathcal{P}$ and the reservoir R from above will be chosen small enough to ensure $\delta_{k-1}(\mathcal{H}'') \geq (\frac{1}{2(k-\ell)} + o(1))|V(\mathcal{H}'')|$. The third auxiliary lemma, the Path-cover Lemma (Lemma 30), asserts that all but $o(n)$ vertices of $\mathcal{H}''$ can be covered by a family of pairwise disjoint ℓ-paths and, moreover, the number of those paths will be constant (independent of n). Consequently, we can connect those paths and $\mathcal{P}$ to form an ℓ-cycle by using exclusively vertices from R. This way we obtain an ℓ-cycle in $\mathcal{H}$, which covers all but the $o(n)$ left-over vertices from $\mathcal{H}''$ and some left-over vertices from R. However, we will ensure that the number of those yet uncovered vertices will be smaller than αn and, hence, we can appeal to the absorption property of $\mathcal{P}$ and obtain a Hamilton ℓ-cycle.

We now state the Absorbing Lemma, the Reservoir Lemma, and the Path-cover Lemma and give the details of the outline above in Section 3.1.3.

3.1.2. Auxiliary lemmas

We start with the Absorbing Lemma. This lemma asserts the existence of a relatively "short", but powerful ℓ-path $\mathcal{P}$ which can "absorb" any small set $U \subseteq V \setminus V(\mathcal{P})$. The proof will be carried out in Section 3.2.

Lemma 28 (Absorbing Lemma)**.** *For all integers $k \geq 3$ and $1 \leq \ell < k/2$ and every $\epsilon > 0$ there exists an $\alpha > 0$ and an n_0 such that for every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geq n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq \epsilon n$ the following holds. There exists an ℓ-path $\mathcal{P} \subset \mathcal{H}$ with $|V(\mathcal{P})| \leq \epsilon^5 n$ such that for all subsets $U \subset V \setminus V(\mathcal{P})$ of size at most $|U| \leq \alpha n$ and $|U| \in (k-\ell)\mathbb{N}$ there exists an ℓ-path $\mathcal{Q} \subset \mathcal{H}$ with $V(\mathcal{Q}) = V(\mathcal{P}) \cup U$ and, moreover, $\mathcal{P}$ and $\mathcal{Q}$ have exactly the same ends.*

The next lemma provides a reservoir $R \subset V$ which we will use to connect short paths to a long one. For a k-uniform hypergraph $\mathcal{H} = (V, E)$, a subset of the vertices $R \subseteq V$ and a $(k-1)$-tuple $S \in \binom{V}{k}$, we denote the set of neighbours of S in R by $N_R(S) = \{v \in R \setminus S \colon S \cup \{v\} \in E\}$ and define $\deg_R(S) = |N_R(S)|$.

Lemma 29 (Reservoir Lemma)**.** *For every integer $k \geq 2$ and every reals d, $\epsilon > 0$ there exists an n_0 such that for every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geq n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq dn$ the following holds. There is a set R of size at most ϵn such that for all $(k-1)$-sets $S \in \binom{V}{k-1}$ we have $\deg_R(S) \geq d\epsilon n/2$.*

Lemma 29 follows directly from the sharp concentration of the hypergeometric distribution.

Proof. For given k, d, and ϵ we choose n_0 sufficiently large and set $q = \lfloor \epsilon n \rfloor$. From $\binom{V}{q}$, the set of all subsets of V with size q, we choose a set R uniformly at random. Now let $S \in \binom{V}{k-1}$ be an arbitrary set of size $(k-1)$ and let $X_S = |N_R(S)|$. Then X_S is hypergeometrically distributed with expectation $\mathbb{E}[X_S] \geq qd \geq 6$. Applying Chernoff's inequality for hypergeometric distribution (Theorem 26) we obtain

$$\mathbb{P}[X_S \leq \lceil dq/2 \rceil] \leq \exp(-dq/30) = \exp(-d\epsilon n/30)$$

Thus, with probability $1 - \binom{n}{k-1}\exp(-d\epsilon n/30) = 1 - o(1)$ every set $S \in \binom{V}{k-1}$ has at least $d\epsilon n/2$ neighbours in R. □

Finally, we state the Path-cover lemma. By an **ℓ-path packing** of a k-uniform hypergraph $\mathcal{H}$ we mean a family of pairwise vertex disjoint ℓ-paths. Then the Path-cover Lemma asserts that a k-uniform hypergraph $\mathcal{H}$ with $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-\ell)} + o(1))|V(\mathcal{H})|$ can be almost perfectly covered by "few" ℓ-paths.

Lemma 30 (Path-cover Lemma)**.** *For all integers $k \geq 3$, all $1 \leq \ell < k/2$ and all γ, $\epsilon > 0$, there exist integers p and n_0 such that for every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geq n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq \left(\frac{1}{2(k-\ell)} + \gamma\right) n$ the following holds. There is an ℓ-path packing of $\mathcal{H}$ consisting of at most p paths, which covers all but at most ϵn vertices of $\mathcal{H}$.*

The proof of Lemma 30 is based on the weak hypergraph regularity lemma and is given in Section 3.3.

3.1.3. Proof of Theorem 4

In this section we give the proof of the main result, Theorem 4. The proof is based on the three auxiliary lemmas introduced in Section 3.1.2 and follows the outline given in Section 3.1.1.

Proof of Theorem 4. Let integers $k \geq 3$ and $1 \leq \ell < k/2$ and a real $\gamma > 0$ be given. Applying the Absorbing Lemma (Lemma 28) for k, ℓ, and $\epsilon_{28} = \gamma/4$ we obtain $\alpha > 0$ and n_{28}. Next we apply the Reservoir Lemma (Lemma 29) for k, ℓ, and $d = 1/(2k)$ and $\epsilon_{29} = \min\{\alpha/2, \gamma/4\}$ we obtain n_{29}. Finally, we apply the Path-cover Lemma (Lemma 30) with $\gamma_{30} = \gamma/2$ and $\epsilon_{30} = \alpha/2$ to obtain p and n_{30}. For n_0 we choose $n_0 = \max\{n_{28}, 2n_{29}, 2n_{30}, 16(p+1)k^2/\epsilon_{29}\}$.

Now let $n \geq n_0$, $n \in (k-\ell)\mathbb{N}$ and let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph on n vertices with

$$\delta_{k-1}(\mathcal{H}) \geq \left(\frac{1}{2(k-\ell)} + \gamma\right) n.$$

Let $\mathcal{P}_0 \subset \mathcal{H}$ be the absorbing ℓ-path guaranteed by Lemma 28 (applied with k, ℓ, and ϵ_{28}). Let F_0^{beg} and F_0^{end} be the ends of $\mathcal{P}_0$ which we may refer to as sets. Note that

$$|V(\mathcal{P}_0)| \leq \epsilon_{28}^5 n < \gamma n/4.$$

Moreover, the path $\mathcal{P}_0$ has the absorption property, i.e. for all $U \subset V \setminus V(\mathcal{P}_0)$ with $|U| \leq \alpha n$ and $|U| \in (k-\ell)\mathbb{N}$

$$\exists\ \ell\text{-path } \mathcal{Q} \subset \mathcal{H} \text{ s.t. } V(\mathcal{Q}) = V(\mathcal{P}_0) \cup U \text{ and } \mathcal{Q} \text{ has the ends } F_0^{\mathrm{beg}} \text{ and } F_0^{\mathrm{end}}. \tag{3.1}$$

Let $V' = (V \setminus V(\mathcal{P}_0)) \cup F_0^{\mathrm{beg}} \cup F_0^{\mathrm{end}}$ and let $\mathcal{H}' = \mathcal{H}[V'] = (V', E(\mathcal{H}) \cap \binom{V'}{k})$ be the induced subhypergraph of $\mathcal{H}$ on V' and note that

$$\delta_{k-1}(\mathcal{H}') \geq (\frac{1}{2(k-\ell)} + 3\gamma/4)n \geq |V'|/(2k) = d|V'|.$$

Due to Lemma 29 we can choose a set $R \subset V' \setminus (F_0^{\mathrm{beg}} \cup F_0^{\mathrm{end}})$ of size at most $\epsilon_{29}|V'| \leq \epsilon_{29} n$ such that

$$|\deg_R(S)| \geq \epsilon_{29}|V'|/(4k) - |F_0^{\mathrm{beg}} \cup F_0^{\mathrm{end}}| \geq \epsilon_{29} n/(8k) \text{ for every } S \in \binom{V'}{k-1}. \tag{3.2}$$

Set $V'' = V \setminus (V(\mathcal{P}_0) \cup R)$ and let $\mathcal{H}'' = \mathcal{H}[V'']$ be the induced subhypergraph of $\mathcal{H}$ on V''. Clearly,

$$\delta_{k-1}(\mathcal{H}'') \geq \left(\frac{1}{2(k-\ell)} + 3\gamma/4 - \epsilon_{29}\right) n \geq \left(\frac{1}{2(k-\ell)} + \gamma/2\right)|V''|.$$

Consequently, Lemma 30 applied to $\mathcal{H}''$ (with γ_{30} and ϵ_{30}) yields an ℓ-path packing of $\mathcal{H}''$ which covers all but at most $\epsilon_{30}|V''| \leq \epsilon_{30} n$ vertices from V'' and consists of at most p paths. We denote the set of the uncovered vertices in V'' by T. Further, let $\mathcal{P}_1, \mathcal{P}_2 \ldots, \mathcal{P}_q$ with $q \leq p$ denote the ℓ-paths of the packing and let F_i^{beg} and F_i^{end} for $i = 1, \ldots, q$ be the ends of the ℓ-path $\mathcal{P}_i$. Recall that the ends of the absorbing ℓ-path $\mathcal{P}_0$ are F_0^{beg} and F_0^{end}. Note that for each $0 \leq i, j \leq q$ we have $|F_i^{\mathrm{end}} \cup F_j^{\mathrm{beg}}| = 2\ell < k$. Thus, for any set $X \subset R$ of size $k - 2\ell - 1$ (X might be empty) we have $\deg_R(F_i^{\mathrm{end}} \cup F_j^{\mathrm{beg}} \cup X) \geq \epsilon_{29} n/(8k) > (p+1)k$ due to (3.2) and the choice of n_0.

Consequently, we can choose $Y_i \subset R \setminus (\bigcup_{0 \leq j < i} Y_j)$ for all $i \in \{0, 1, \ldots, q\}$, such that $F_i^{\mathrm{end}} \cup Y_i \cup F_{(i+1) \bmod (q+1)}^{\mathrm{beg}}$ is an edge in $E(\mathcal{H}) \setminus \bigcup_{i=0}^{q} E(\mathcal{P}_i)$. Hence, we can connect all paths $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_q$, and $\mathcal{P}_0$ to an ℓ-cycle $\mathcal{C} \subseteq \mathcal{H}$.

Let $U = V \setminus V(\mathcal{C})$ be the set of vertices not covered by the ℓ-cycle $\mathcal{C}$. Since $U \subseteq R \cup T$ we have $|U| \leq (\epsilon_{30} + \epsilon_{29})n \leq \alpha n$. Moreover, since $\mathcal{C}$ is an ℓ-cycle and $n \in (k-\ell)\mathbb{N}$ we have $|U| \in (k-\ell)\mathbb{N}$. Thus, using the absorption property of $\mathcal{P}_0$ (see (3.1)) we can replace the subpath $\mathcal{P}_0$ in $\mathcal{C}$ by a path $\mathcal{Q}$ (since $\mathcal{P}_0$ and $\mathcal{Q}$ have the same ends) and since $V(\mathcal{Q}) = V(\mathcal{P}_0) \cup U$ the resulting ℓ-cycle is a Hamilton ℓ-cycle of $\mathcal{H}$. $\square$

3.2. The Absorbing Lemma

In this section we prove Lemma 28, the Absorbing Lemma. Roughly speaking, "absorption" stands for a local extension of a given structure, which preserves the global structure. For ℓ-paths, e.g., we want to insert a set S of vertices to an existing ℓ-path, i.e. to "absorb" S, in such a way that the new object is again an ℓ-path which, moreover, has the same ends.

Definition 31. *Let $k \geq 3$ and $1 \leq \ell < k/2$ be integers and $\mathcal{H} = (V, E)$ be a k-uniform hypergraph. We say an ℓ-path with three edges $\mathcal{P} \subseteq \mathcal{H}$ and ends F^{beg} and F^{end} is an* ***absorbing path*** *for a $(k-\ell)$-set $S \in \binom{V \setminus V(\mathcal{P})}{k-\ell}$, if there exists an ℓ-path $\mathcal{Q}$ with four edges with the same ends F^{beg} and F^{end} and $V(\mathcal{Q}) = V(\mathcal{P}) \cup S$.*

Moreover, if $\mathcal{P}$ is an absorbing path for S with ends F^{beg} and F^{end}, then we call the t-set $T = V(\mathcal{P}) \in \binom{V \setminus S}{t}$ with $t = 3(k-\ell)+\ell$ an ***absorbing t-tuple for*** *S with ends F^{beg} and F^{end}.*

Given that an absorbing ℓ-path $\mathcal{P}$ for S was part of some long ℓ-path, then the local change of absorbing S does not destroy the long path since the ends of $\mathcal{P}$ and $\mathcal{Q}$ are the same. Clearly, for any fixed $(k-\ell)$-set S there are at most $O(n^t)$ absorbing t-tuples. The following proposition, however, says that this bound is achieved up to a constant factor when the minimum $(k-1)$-degree of $\mathcal{H}$ is linear in n.

Proposition 32. *Let $k \geq 3$, $1 \leq \ell < k/2$, and $\epsilon > 0$. Furthermore, let $\mathcal{H}$ be a k-uniform hypergraph on $n \geq 6k/\epsilon$ vertices with $\delta_{k-1}(\mathcal{H}) \geq \epsilon n$. Then for every $(k-\ell)$-set $S \in \binom{V}{k-\ell}$ there are at least $\epsilon^5 \binom{n}{t}/(2^{5+3k}k^4)$ absorbing t-tuples $T \in \binom{V \setminus S}{t}$ with $t = 3(k-\ell)+\ell$.*

We postpone the proof of Proposition 32 and we first deduce Lemma 28 from it.

Proof of Lemma 28. Let $k \geq 3$, $1 \leq \ell < k/2$, and $\epsilon > 0$ be given. We set $t = 3(k-\ell)+\ell$ and fix auxiliary constants

$$\zeta = \frac{\epsilon^5 (t-2\ell)!}{2^{6+3k}k^4 t!} \quad \text{and} \quad \rho = \frac{\zeta}{16t^2} < \frac{\epsilon^5}{8t}\,.$$

Finally we set

$$\alpha = \zeta\rho/4$$

and let $n_0 \geq 6k/\epsilon$ be sufficiently large.

Suppose $\mathcal{H} = (V, E)$ is a k-uniform hypergraph on $n \geq n_0$ vertices which satisfies $\delta_{k-1}(\mathcal{H}) \geq \epsilon n$. Note that in Proposition 32 the ends of the absorbing t-tuples are not specified yet. This we now do by taking the ends $F_T^{beg}, F_T^{end} \subset T$ of an arbitrary t-set $T \in \binom{V}{t}$ uniformly at random, i.e. with probability $(t-2\ell)!/t!$ a given pair of disjoint, ordered ℓ-tuples will become the ends of T. Hence, due to Proposition 32, the expected number of absorbing t-tuples (now with distinguished ends) for a fixed $(k-\ell)$-set $S \in \binom{V}{k-\ell}$ is at least $2\zeta\binom{n}{t}$. Applying Chernoff's inequality (Theorem 26) we derive that there is a choice of ends for all t-sets which yields at least $\zeta\binom{n}{t}$ absorbing t-tuples with distinguished ends for all $(k-\ell)$-sets. We fix such a choice and for a fixed

$(k-\ell)$-set $S \in \binom{V}{k-\ell}$ let $\mathcal{T}(S)$ denote the set of the absorbing t-tuples T for S with ends F_T^{beg} and F_T^{end} according to this choice. Thus, we have $|\mathcal{T}(S)| \geq \zeta\binom{n}{t}$ for all $S \in \binom{V}{k-\ell}$.

Next we pick a family $\mathscr{T} \subseteq \binom{V}{t}$ randomly, where each t-tuple $T \in \binom{V}{t}$ is included in $\mathscr{T}$ independently with probability $p = \rho n/\binom{n}{t}$. Hence, we have

$$\mathbb{E}\left[|\mathscr{T}|\right] = \rho n \quad \text{and} \quad \mathbb{E}\left[|\mathscr{T} \cap \mathcal{T}(S)|\right] \geq \zeta \rho n \quad S \in \binom{V}{k-\ell}.$$

From Chernoff's inequality (Theorem 26) we infer that with probability $1 - o(1)$

$$|\mathscr{T}| \leq 2\rho n \tag{3.3}$$

and

$$|\mathscr{T} \cap \mathcal{T}(S)| \geq \zeta\rho n/2 \text{ for all } S \in \tbinom{V}{k-\ell}. \tag{3.4}$$

Furthermore, let $I(\mathscr{T})$ denote the number of intersecting t-tuples in $\mathscr{T}$, i.e. the number of pairs T and $T' \in \mathscr{T}$ such that $T \cap T' \neq \emptyset$. Then

$$\mathbb{E}\left[I(\mathscr{T})\right] \leq t\binom{n}{t}\binom{n}{t-1} \times p^2 = \frac{t^2\rho^2 n^2}{n-t+1} \leq 2t^2\rho^2 n = \zeta\rho n/8$$

due to the choice of ρ, and using Markov's inequality we conclude that with probability at least $1/2$

$$I(\mathscr{T}) \leq \zeta\rho n/4. \tag{3.5}$$

In particular, the properties (3.3), (3.4), and (3.5) hold simultaneously with positive probability for the randomly chosen family $\mathscr{T}$. So, let $\mathscr{T}'$ be a family satisfying (3.3), (3.4), and (3.5). By deleting all intersecting t-tuples from $\mathscr{T}'$ and all those t-tuples which do not absorb any $S \in \binom{V}{k-\ell}$ we obtain a family $\mathscr{T}'' \subset \mathscr{T}'$ of pairwise disjoint t-tuples of size at most $2\rho n$ which, due to (3.4), (3.5), and the choice of α, satisfies

$$|\mathscr{T}'' \cap \mathcal{T}(S)| \geq \zeta\rho n/4 = \alpha n \tag{3.6}$$

for all $S \in \binom{V}{k-\ell}$.

Lastly, we want to connect the t-tuples in $\mathscr{T}''$ to create an ℓ-path. To this end, let $\mathscr{T}'' = \{T_1, \ldots, T_r\}$ for some $r \leq 2\rho n$ and let F_i^{beg} and F_i^{end} be the ends of T_i. Since every T_i (with its chosen ends F_i^{beg} and F_i^{end}) absorbs at least one $(k-\ell)$-set, the induced hypergraph $\mathcal{H}[T_i]$ must contain an ℓ-path $\mathcal{P}_i$ with three edges and ends F_i^{beg} and F_i^{end}. For $i = 1, \ldots, r-1$ observe further that $|F_i^{\mathrm{end}} \cup F_{i+1}^{\mathrm{beg}}| = 2\ell$ and, hence, for any V_i of size at least $n - 4\rho n t$ and any $Y \in \binom{V_i}{k-2\ell-1}$ we know

$$|N_{V_i}(F_i^{\mathrm{end}} \cup F_{i+1}^{\mathrm{beg}} \cup Y)| \geq \epsilon n - 4\rho n t > 0\,.$$

Thus, we can choose $X_i \in N_{V_i}(F_i^{\mathrm{end}} \cup F_{i+1}^{\mathrm{beg}})$ to connect $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ through the edge $F_i^{\mathrm{end}} \cup X_i \cup F_{i+1}^{\mathrm{beg}}$. Starting with the set $V_1 = V(\mathcal{H}) \setminus \bigcup_{T \in \mathscr{T}''} V(T)$ of size $|V_1| \geq n - 2\rho n t$

we connect $\mathcal{P}_1$ and $\mathcal{P}_2$. We continue by induction. So suppose for some $i < r$ we chose sets $X_1, \ldots, X_{i-1}$ and used them to connect the ℓ-paths $\mathcal{P}_1, \ldots, \mathcal{P}_i$ to one ℓ-path. With $V_i = V_1 \setminus (\bigcup_{j=1}^{i-1} X_j)$ which has size at least $n - 2\rho n t - i(k-2\ell) > n - 4\rho n t$ and by the observation from above we connect $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ by choosing $X_i \in N_{V_i}(F_i^{end} \cup F_{i+1}^{\mathrm{beg}})$. Consequently, we can connect all ℓ-paths $\mathcal{P}_1, \ldots, \mathcal{P}_r$ to one ℓ-path $\mathcal{P}$ containing at most $4\rho n t \leq \epsilon^5 n$ vertices.

Finally, suppose $U \subset V \setminus V(\mathcal{P})$ with $|U| \leq \alpha n$ and $|U| \in (k-\ell)\mathbb{N}$. Then we partition U into $q \leq \alpha n/(k-\ell)$ pairwise disjoint sets $S_1, \ldots, S_q$ each of size $(k-\ell)$. But since (3.6) holds, we can absorb each S_i, $i = 1, \ldots, q$ one by one taking an unused absorbing t-tuple $T_i \in \mathscr{T}'' \cap \mathscr{T}_S$ for each S_i. This way we obtain an ℓ-path $\mathcal{Q}$ which covers exactly the vertices in $V(\mathcal{P}) \cup U$ and the lemma follows. $\square$

We complete the proof of Lemma 28 by proving Proposition 32. To this end we need the notion of a "neighbourhood" of a set $S \subset V(\mathcal{H})$ in a set $U \subset V(\mathcal{H})$. This is given by $N_U(S) = \{X \subset U \setminus S \colon S \cup X \in E(\mathcal{H})\}$.

Proof of Proposition 32. Let $S \in \binom{V}{k-\ell}$ be an arbitrary set of size $k-\ell$ and set $V_0 = V \setminus S$. In the following we will choose pairwise disjoint sets A, B_1, B_2, C, D_1, and D_2 whose union forms an absorbing t-tuple for S.

We start by choosing $A \in \binom{V_0}{k-2\ell}$ arbitrarily. Then the number of choices for A is

$$\binom{n-k+\ell}{k-2\ell}. \tag{3.7}$$

Set $V_1 = V_0 \setminus A$ and split $S \dot\cup A = Z_1 \dot\cup L \dot\cup Z_2$ in an arbitrary way such that $|L| = \ell$ and $|Z_1| = |Z_2| = k - 2\ell$. Next we choose $B_1 \in N_{V_1}(Z_1 \cup L)$ and $B_2 \in N_{V_2}(Z_2 \cup L)$ where $V_2 = V_1 \setminus B_1$. To compute the number of choices for B_1 and B_2 note that $|V_2| = n - 2k + 3\ell$, $|V_3| = n - 2k + 2\ell$ and for every set $X_i \in \binom{V_i}{\ell-1}$, $i = 1, 2$, we know that $\deg_{\mathcal{H}}(Z_i \cup L \cup X_i) \geq \epsilon n$ thus $N_{V_i}(Z_i \cup L \cup X_i)$ has size at least $\epsilon n - 2k \geq \epsilon n/2$, since $n \geq 4k/\epsilon$. This way we count each possible B_i in ℓ ways. Consequently, the number of choices for B_1 and B_2, i.e. $|N_{V_2}(Z_1 \cup L)| \times |N_{V_3}(Z_2 \cup L)|$ is at least

$$\left(\frac{\epsilon n}{2\ell}\right)^2 \binom{n-2k+3\ell}{\ell-1} \binom{n-2k+2\ell}{\ell-1}. \tag{3.8}$$

Next, set $V_3 = V_2 \setminus B_2$ and for $i = 1, 2$ let $B_i' \subset B_i$ of size $|B_i'| = |B_i| - 1$ (thus, B_i' may be empty if $\ell = 1$). We choose the set $C \in N_{V_3}(A \cup B_1' \cup B_2')$. Since $|V_3| = n - 2k + \ell$ by arguing as above for B_1 and B_2 we conclude that the number of choices for C is at least

$$\frac{1}{2}(n - 2k + \ell)(\epsilon n - 2k) \geq \frac{\epsilon n^2}{8}. \tag{3.9}$$

We set $V_4 = V_5 \setminus C$ and for $C = \{v_1, v_2\}$ we choose $D_1 \in N_{V_4}(B_1 \cup \{v_1\})$. With $V_5 = V_4 \setminus D_1$ take $D_2 \in N_{V_5}(B_2 \cup \{v_2\})$. Note that $|V_5| = n - 2k + \ell - 2$, $|V_6| = n - 3k - 1$ and $|B_i \cup \{v_i\}| = \ell + 1$. Thus, again, by arguing as for B_1, B_2 we derive that the number of choices for D_1 and D_2 is at least

$$\left(\frac{\epsilon n}{2(k-\ell-1)}\right)^2 \binom{n-2k+\ell-2}{k-\ell-2}\binom{n-3k-1}{k-\ell-2}. \tag{3.10}$$

For given S let

$$T = A\dot{\cup}B_1\dot{\cup}B_2\dot{\cup}C\dot{\cup}D_1\dot{\cup}D_2$$

and note that

$$|T| = |A| + |B_1| + |B_2| + |C| + |D_1| + D_2| = 3(k-\ell) + \ell = t\,.$$

Combining (3.7), (3.8), (3.9), and (3.10) we obtain that the number of choices for T chosen as above for a given set S is at least

$$\frac{\epsilon^5}{2^7\ell^2k^2}\binom{n-k+\ell}{t} \geq \frac{\epsilon^5}{2^{7+t}\ell^2k^2}\binom{n}{t} \geq \frac{\epsilon^5}{2^{5+3k}k^4}\binom{n}{t}.$$

We now verify that T is indeed an absorbing t-tuple for S. For that we "reorder" the vertices of T and observe that

$$T = D_1\dot{\cup}B_1\dot{\cup}\{v_1\}\dot{\cup}A\dot{\cup}\{v_2\}\dot{\cup}B_2\dot{\cup}D_2\,.$$

Note that

$$E_1 = D_1\dot{\cup}B_1\dot{\cup}\{v_1\}\,,\quad G = B_1'\dot{\cup}\{v_1\}\dot{\cup}A\dot{\cup}\{v_2\}\dot{\cup}B_2'\,,\quad \text{and}\quad E_2 = \{v_2\}\dot{\cup}B_2\dot{\cup}D_2$$

are edges in $\mathcal{H}$ and that they form an ℓ-path $\mathcal{P}$, since $|E_i \cap G| = |B_i' \cup \{v_i\}| = \ell$, for $i = 1, 2$. For the ends of this path we could fix any ordering of any ℓ-set from D_i. Moreover, the sets

$$G_1 = B_1\dot{\cup}Z_1\dot{\cup}L \quad \text{and} \quad G_2 = L\dot{\cup}Z_2\dot{\cup}B_2$$

are also edges of $\mathcal{H}$ and E_1, G_1, G_2, E_2 forms an ℓ-path $\mathcal{Q}$ with $V(\mathcal{Q}) = S\dot{\cup}T$, since $|G_i \cap E_i| = |B_i| = \ell$, for $i = 1, 2$ and $|G_1 \cap G_2| = |L| = \ell$. The ends of this ℓ-path can be chosen to coincide with the ends of $\mathcal{P}$, since $D_i \cap G_i = \emptyset$ for $i = 1, 2$.

This proves that any set T chosen as above is indeed an absorbing t-tuple for S. $\square$

3.3. The Path-cover Lemma

In this section we prove the Path-cover Lemma, Lemma 30. The proof combines the techniques in [96] and [79].

3.3.1. Almost perfect $\mathcal{F}_{k,\ell}$-packings

First we show that an n-vertex, k-uniform hypergraph $\mathcal{H}$ satisfying the minimum degree condition $\delta_{k-1}(\mathcal{H}) \geq n/(2(k-\ell))$ contains a $\mathcal{F}_{k,\ell}$-packing which covers all but $o(n)$

vertices of $\mathcal{H}$, where $\mathcal{F}_{k,\ell}$ is defined as follows.

Definition 33. *For positive integers k and ℓ let $\mathcal{F}_{k,\ell}$ be the k-uniform hypergraph on $2(k-\ell)(k-1)$ vertices such that its vertex set consists of pairwise disjoint sets $A_1, A_2, \dots, A_{2k-2\ell-1}, B$, each of size $k-1$, and its edge set consists of all sets $A_i \cup \{b\}$ where $i \in [2k-2\ell-1]$ and $b \in B$.*

Kühn and Osthus [79] considered $\mathcal{F}_{3,1}$-packings, i.e. families of pairwise vertex disjoint copies of $\mathcal{F}_{3,1}$. The proof of the $\mathcal{F}_{k,\ell}$-packing lemma, Lemma 34, follows their approach.

Lemma 34 ($F_{k,\ell}$-packing Lemma)**.** *For all integers $k \geq 3$ and $1 \leq \ell < k$ and every $\epsilon > 0$ there exists an n_0 such that for every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geq n_0$ vertices the following holds.*

If $\deg_{k-1}(S) \geq n/(2(k-\ell))$ for all but at most ϵn^{k-1} sets $S \in \binom{V}{k-1}$, then $\mathcal{H}$ contains a $F_{k,\ell}$-packing covering all but at most $(5\epsilon)^{1/(k-1)}n$ vertices.

Proof. For given k, ℓ, and ϵ we choose n_0 large enough and set $\delta = (5\epsilon)^{1/(k-1)}$. Suppose $\mathcal{A} = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{i_0}\}$ is a largest $\mathcal{F}_{k,\ell}$-packing leaving the vertex set $X \subset V$ of size $|X| \geq \delta n$ uncovered.

From the condition on the degree for $\mathcal{H}$ we first show the following.

Claim 35. *There is a family $\mathcal{B}$ of size $\delta n/(2k^k)$ which consists of mutually disjoint $(k-1)$-sets $S \in \binom{X}{k-1}$ such that $\deg(S) \geq n/(2(k-\ell))$ and $|N_X(S)| \leq \delta n/(4k)$ for all $S \in \mathcal{B}$.*

Proof. The claim follows from a probabilistic argument. First we split X into two parts $X = X_1 \dot\cup X_2$ by choosing $X_2 \subset X$ of size $|X|/(2k)$ uniformly at random. Thereafter, we take a family $\mathcal{S}$ consisting of $\delta n/k^k$ pairwise disjoint sets $S \in \binom{X_1}{k-1}$ from X_1 such that $\deg(S) \geq n/(2(k-\ell))$. Such a family exists indeed, since the number of $(k-1)$-sets with degree falling below $n/(2(k-\ell))$ is at most ϵn^{k-1} and due to the choice of δ

$$\binom{|X_1|}{k-1} - \epsilon n^{k-1} \geq (k-1)\frac{\delta n}{k^k}\binom{|X_1|}{k-2}.$$

Next, we claim that at least half, i.e. $\delta n/(2k^k)$, of the chosen $(k-1)$-sets S_i must satisfy $|N_X(S_i)| \leq \delta n/(4k)$ since otherwise the $\mathcal{F}_{k,\ell}$-packing $\mathcal{A}$ was not largest possible. For a contradiction, let $\mathcal{S}' \subset \mathcal{S}$ denote the set of the chosen $S_i \in \mathcal{S}$ such that $|N_X(S)| > \delta n/(4k)$ and suppose $\mathcal{S}' = \{S_1, \dots, S_r\}$ has size $r \geq \delta n/(2k^k)$.

For any $(k-1)$-sets $S \in \binom{X_1}{k-1}$ with $|N_X(S)| > |X|/(4k)$ let $Y_S = |N_{X_2}(S)|$ denote the size of its neighbourhood in X_2. Then Y_S has hypergeometric distribution with mean $\mathbb{E}[Y_S] \geq (|X|/(4k)) \times (1/(2k)) \geq \delta n/(8k^2)$ and applying Chernoff's inequality (Theorem 26) we conclude

$$p = \mathbb{P}\left[|Y_S| \leq \delta n/(16k^2)\right] \leq \exp\{-\delta n/(100k^2)\}.$$

Hence, with a probability at least $1 - \binom{|X|}{k-1}p = 1 - o(1)$ all sets $S \in \binom{X}{k-1}$ with $|N_X(S)| > |X|/(4k)$ also satisfy $|N_{X_2}(S)| \geq n/(16k^2)$. In particular, almost surely

$|N_{X_2}(S)| \geq n/(16k^2)$ is satisfied for all $S \in \mathcal{S}'$ and we assume that this indeed happens for the decomposition $X = X_1 \dot\cup X_2$ we have chosen. Now consider the auxiliary bipartite graph G with vertex classes $\mathcal{S}'$ and X_2 and with $\{S, v\}$ being an edge if and only if $S \cup \{v\} \in \mathcal{H}$. Then every S has at least $\delta n/(16k^2)$ neighbours, thus, by the result of Kövari, Sós, and Turán (Theorem 23) the graph G contains a $K_{k,k-1}$. However, this $K_{k,k-1}$ in G corresponds to a copy of $\mathcal{F}_{k,\ell}$ in $\mathcal{H}$, which is a contradiction to $\mathcal{A}$ being the largest $\mathcal{F}_{k,\ell}$-packing. □

Continuing the proof of Lemma 34, we fix a family $\mathcal{B} = \{S_1, \ldots, S_q\}$, $q = \delta n/(2k^k)$ as stated in the claim above. For a set $S_i \in \mathcal{B}$ we say that an element $\mathcal{F}$ from the $\mathcal{F}_{k,\ell}$-packing $\mathcal{A}$ is *good* for S_i if $\mathcal{F}$ contains at least k neighbours of S_i, i.e. $|N_{V(\mathcal{F})}(S_i)| \geq k$. With n_i denoting the number of good $\mathcal{F} \in \mathcal{A}$ for S_i and $t = 2(k-\ell)(k-1)$ we conclude from the condition on $\deg(S_i)$ that

$$\frac{n}{2(k-\ell)} \leq \deg(S_i) \leq (k-1)\frac{(1-\delta)n}{t} + tn_i + \frac{\delta n}{4k} \tag{3.11}$$

$$\leq \frac{(1-\delta/2)n}{2(k-\ell)} + tn_i. \tag{3.12}$$

From this we infer that $n_i \geq \delta n/(8k^3) = n^*$. Next, we want to count all those pairs $(S, \mathcal{T})$ with $\mathcal{T} = \{\mathcal{F}^1, \ldots, \mathcal{F}^{k-1}\} \in \binom{\mathcal{A}}{k-1}$ such that each $\mathcal{F} \in \mathcal{T}$ is good for $S \in \mathcal{B}$. Such a pair $(S, \mathcal{T})$ we call a *good pair* and the number of good pairs is at least $|\mathcal{B}|\binom{n^*}{k-1} \geq (\delta n)^k/(8k^5)^k$. Thus by averaging we infer that there must be a $\mathcal{T}$ and at least $\delta^k n/(8k^5)^k$ sets $S_i \in \mathcal{B}$ such that $(S_i, \mathcal{T})$ are a good pairs.

Hence, there is a $\mathcal{B}' \subseteq \mathcal{B}$ containing at least $(\delta^k n/(8k^5)^k)/\binom{2(k-\ell)(k-1)}{k}^{k-1}$ pairwise disjoint $(k-1)$-sets S from $\mathcal{B}$ and for every $j = 1, \ldots k-1$ there exist k vertices $v_1^j, \ldots, v_k^j$ in $\mathcal{F}^j$ such that

$$S \cup \{v_1^j\}, \ldots, S \cup \{v_k^j\} \in E(\mathcal{H}) \text{ for every } S \in \mathcal{B}' \text{ and } j = 1, \ldots k-1\,.$$

Since $(\delta^k n/(8k^5)^k)/\binom{2(k-\ell)(k-1)}{k}^{k-1} \geq (2(k-\ell)-1)k$ for sufficiently large n, we can select k families mutually disjoint families $\{S_1^i, \ldots, S_{2k-2\ell-1}^i\} \subseteq \mathcal{B}'$ for $i = 1, \ldots, k$. Now for every $i = 1, \ldots, k$ the set

$$\{S_p^i \cup \{v_i^j\} \colon\ p = 1, \ldots, 2k-2\ell-1,\ j = 1, \ldots, k-1\}$$

is the edge set of a copy of $\mathcal{F}_{k,\ell}$ and we obtain k mutually disjoint copies of $\mathcal{F}_{k,\ell}$ this way. Replacing the $(k-1)$-copies $\mathcal{F}^1, \ldots, \mathcal{F}^{k-1}$ by those k copies enlarges the $\mathcal{F}_{k,\ell}$-packing $\mathcal{B}$, which is a contradiction. □

3.3.2. Almost perfect path-packings in regular k-tuples

In this section we show that (ϵ, d)-regular k-tuples $(V_1, \ldots, V_k)$ can be almost perfectly covered by ℓ-paths.

Definition 36. *Suppose $\mathcal{H}$ is a k-uniform, k-partite hypergraph with partition classes $V_1, V_2, \ldots, V_k$. Then we call an ℓ-path $\mathcal{P} \subset \mathcal{H}$ with t edges $(E_1, \ldots, E_t)$* ***canonical*** *with respect to $(V_1, V_2, \ldots, V_k)$ if*

$$E_i \cap E_{i+1} \subset \bigcup_{j \in [\ell]} V_j \quad \text{or} \quad E_i \cap E_{i+1} \subset \bigcup_{j \in [k] \setminus [k-\ell]} V_j$$

for all $i = 1, 2, \ldots, t-1$.

Further, we say that V_i is in ***end position*** *if it is one of the first or the last ℓ elements in the ordering, i.e. $i \in [\ell] \cup \{k - \ell + 1, \ldots, k\}$, whereas V_i is in* ***middle position*** *if $i \in \{\ell + 1, \ldots, k - \ell\}$.*

Remark 37. *Let t be a odd number. If $\mathcal{P}$ with t edges is a canonical path with respect to $(V_1, \ldots, V_k)$ and $n_i = |V(\mathcal{P}) \cap V_i|$, then*

$$n_i = \begin{cases} (t+1)/2 & \text{if } V_i \text{ is in end position,} \\ t & \text{if } V_i \text{ is in middle position.} \end{cases}$$

The following proposition was essentially proven in [96].

Proposition 38. *Suppose $\mathcal{H}$ is a k-partite, k-uniform hypergraph with the partition classes $V_1, V_2, \ldots, V_k$, $|V_i| = m$ for all $i \in [k]$, and $|E(\mathcal{H})| \geq dm^k$. Then there exists a canonical ℓ-path in $\mathcal{H}$ with respect to $(V_1, \ldots V_k)$ with $t > dm/(2(k - \ell))$ edges.*

Proof. First we consider all possible ends of a canonical ℓ-path $\mathcal{P}$, i.e. all ℓ-sets $L \subset V(\mathcal{H})$ such that

$$|L \cap V_i| = 1 \quad \text{either for all } i \in [\ell] \text{ or for all } i \in [k] \setminus [k - \ell].$$

For a possible end L such that $\deg(L) = |\{E \in \mathcal{H} \colon L \subset E\}| < dm^{k-\ell}/2$ we delete all edges from the current hypergraph which contain L. We keep doing this until every possible end L satisfies $\deg(L) = 0$ or $\deg(L) \geq dm^{k-\ell}/2$ in the present hypergraph. Note that the number of edges we have deleted is less than $2m^\ell \times dm^{k-\ell}/2 = dm^k$, hence, the final hypergraph $\mathcal{H}'$ is non-empty. We pick a maximal canonical ℓ-path $\mathcal{P} \subset \mathcal{H}'$ with respect to $(V_1, \ldots, V_k)$ which has $t \geq 1$ edges and let the ℓ-set L denote one end of $\mathcal{P}$. Since L is contained in an edge in $\mathcal{H}'$ we know that $\deg(L) \geq dm^{k-\ell}/2$. On the other hand, every edge in $\mathcal{H}'$ which contains L must intersect $V(\mathcal{P}) \setminus L$ since $\mathcal{P}$ is maximal. Thus, we have

$$\frac{dm^{k-\ell}}{2} \leq \deg(L) < \left((k - 2\ell)t + \ell\frac{(t+1)}{2}\right) m^{k-\ell-1} \leq (k - \ell)tm^{k-\ell-1}.$$

This yields $t > dm/(2(k - \ell))$. $\square$

We want to use Proposition 38 to cover a ϵ-regular tuple $(V_1, \ldots, V_k)$ by ℓ-paths which intersect $V_1, \ldots, V_{k-1}$ equally and which, moreover, intersect V_k almost as little as possible.

Lemma 39. *For all integers $k \geq 3$, $1 \leq \ell < k/2$, and all $d, \beta > 0$ there exist $\epsilon > 0$, p and m_0 such that for all $m > m_0$ the following holds. Suppose $\mathcal{V} = (V_1, V_2, \ldots, V_k)$ is an (ϵ, d)-regular k-tuple with $|V_i| = (2k - 2\ell - 1)m$ for all $i \in [k-1]$ and $|V_k| = (k-1)m$. Then there is a family consisting of at most p pairwise vertex disjoint ℓ-paths which cover all but at most βm vertices of $\mathcal{V}$.*

Proof. Let k, ℓ, d, and β be given. We choose $\epsilon = \min\{d/2, \beta/(7k^2), 1/k!\}$, $p = 2k/\epsilon^2$, and $m_0 > 2\epsilon^{-3}$ sufficiently large. Suppose $\mathcal{V} = (V_1, \ldots, V_k)$ is an (ϵ, d)-regular tuple as stated in the lemma. We choose t to be the largest odd number satisfying $t \leq \lfloor \epsilon^2 km/(k-\ell) \rfloor$ and we want to cover $\mathcal{V}$ by ℓ-paths each having t edges. To this end, let S_{k-1} denote the symmetric group and for each permutation $\tau \in S_{k-1}$ let

$$\mathcal{V}(\tau) = (V_{\tau(1)}, V_{\tau(2)}, \ldots, V_{\tau(k-1)}, V_k).$$

Let p_0 denote the maximal integer for which there exists a family of pairwise disjoint ℓ-paths with exactly t edges each, such that every ℓ-path is canonical with respect to some $\mathcal{V}(\tau)$, $\tau \in S_{k-1}$, and for every $\tau \in S_{k-1}$ there are either exactly p_0 or $p_0 + 1$ paths in this family which are canonical with respect to $\mathcal{V}(\tau)$. Among those families let $\mathscr{P}_{p_0}$ be one with maximal cardinality and for each $\tau \in S_{k-1}$ for which there are $p_0 + 1$ canonical ℓ-paths with respect to $\mathcal{V}(\tau)$ in $\mathscr{P}_{p_0}$ we remove one of those paths to obtain $\mathscr{P} \subset \mathscr{P}_{p_0}$ with size $|\mathscr{P}| = p_0(k-1)!$. We will prove that $\mathscr{P}$ is the family of ℓ-paths regquired in the lemma.

For a family $\mathscr{P}'$ of paths let $V(\mathscr{P}') = \bigcup_{\mathcal{P} \in \mathscr{P}'} V(\mathcal{P})$ and we claim that there is an $\tilde{r} \in [k]$ such that $|V_{\tilde{r}} \setminus V(\mathscr{P}_{p_0})| < 2k\epsilon m$. In the opposite case we pick $W_r \subset V_r \setminus V(\mathscr{P}_{p_0})$ with size $|W_r| = 2k\epsilon m$ for all $r \in [k]$ and from regularity of $(V_1, \ldots, V_k)$ and $W_r \subset V_r$ we derive that

$$e(W_1, \ldots, W_k) \geq (d - \epsilon)(2k\epsilon m)^k.$$

Since $d \geq 2\epsilon$ we know from Proposition 38 that for any $\tau \in S_{k-1}$ there is a canonical ℓ-path with respect to $(W_{\tau(1)}, \ldots, W_{\tau(k-1)}, W_k)$ consisting of more than $\epsilon^2 km/(k-\ell) \geq t$ edges. (Note that these ℓ-paths are not necessarily disjoint for different τ.) However, we get a contradiction either to the maximality of p_0 or to the maximality of $|\mathscr{P}_{p_0}|$.

Thus, with $U_r = V_r \cap V(\mathscr{P})$ for all $r \in [k]$, we derive that there exists an $\tilde{r} \in [k]$ such that

$$|U_{\tilde{r}}| \geq |V_{\tilde{r}}| - |\mathscr{P}_{p_0} \setminus \mathscr{P}|t - 2k\epsilon m \geq |V_{\tilde{r}}| - 3k\epsilon m,$$

since $|\mathscr{P}_{p_0} \setminus \mathscr{P}| \leq (k-1)!$, $t \leq \epsilon^2 km/(k-\ell)$, and $\epsilon \leq 1/k!$.

From the above we want to derive that

$$|U_r| \geq |V_r| - 7k\epsilon m \quad \text{for all } r \in [k] \tag{3.13}$$

which would imply the lemma, since $\epsilon \leq \beta/(7k^2)$.

To this end, note first that canonical ℓ-paths with t edges intersect sets in middle position in exactly t vertices, whereas sets in end positions are intersected in $(t+1)/2$

vertices (see Remark 37). Hence, for all $r \in [k-1]$ we have

$$\begin{aligned}|U_r| &= p_0\Big[(k-2\ell)(k-2)!t + (2\ell-1)(k-2)!(t+1)/2\Big] \\ &= p_0\Big[(2k-2\ell-1)(k-2)!(t+1)/2 - (k-2\ell)(k-2)!\Big]\end{aligned}$$

and

$$|U_k| = p_0(k-1)!(t+1)/2.$$

Suppose $\tilde{r} \neq k$ then $|U_r| = |U_{\tilde{r}}| \geq |V_{\tilde{r}}| - 3k\epsilon m$ for all $r \in [k-1]$ and

$$p_0 \geq \frac{2}{(t+1)}\frac{|U_{\tilde{r}}|}{(2k-2\ell-1)(k-2)!}.$$

However, this implies

$$|U_k| \geq \frac{(k-1)|U_{\tilde{r}}|}{2k-2\ell-1} \geq (k-1)m - 3k\epsilon m = |V_k| - 3k\epsilon m.$$

On the other hand, if $\tilde{r} = k$ then

$$p_0 = \frac{2}{(t+1)}\frac{|U_k|}{(k-1)!}$$

from which we derive

$$|U_r| \geq (2k-2\ell-1)m - 7k\epsilon m = |V_k| - 7k\epsilon m$$

due to $m \geq m_0 \geq 2\epsilon^{-3}$. In both cases, we obtain (3.13).

To finish the proof note that $p_0(k-1)!(t+1)/2 \leq |V_k| = (k-1)m$ from which we infer $|\mathscr{P}| \leq 2k/\epsilon^2 = p$. $\square$

3.3.3. Proof of the Path-cover Lemma

In this section we prove the Lemma 30.

Proof of Lemma 30. Given k, ℓ with $k > 2\ell$ and $\gamma, \epsilon > 0$. We apply Lemma 39 with k, ℓ, $d = \gamma/6$ and $\beta = \epsilon/3$ to obtain ϵ_{39}, p_{39} and m_{39} and subsequently apply Lemma 34 with k, ℓ, $\epsilon_{34} = (\epsilon/3)^{(k-1)}/5$ to obtain n_{34}. Finally, we apply Theorem 19 with k and

$$\epsilon_{19} = \frac{1}{2}\min\left\{\frac{\gamma^2}{16}, \frac{\gamma}{24k}, \epsilon_{34}^2, \frac{\epsilon_{39}}{2k}\right\} \quad \text{and} \quad t_{19} = \max\left\{n_{34}, \frac{16k}{\epsilon_{19}}\right\}$$

to obtain T_{19} and n_{19}. Let $p = T_{19}p_{39}$ and $n_0 \geq \max\{2k^2 T_{19}/\epsilon_{19}, n_{19}\}$ sufficiently large.

For a hypergraph $\mathcal{H}$ on $n \geq n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-\ell)} + \gamma)n$ we apply the weak hypergraph regularity lemma (Theorem 19) with k, ϵ_{19} and t_{19}. By possibly moving at most $t(2k-2\ell-1)(k-1) < \epsilon_{19}n$ vertices to V_0 we obtain an $2\epsilon_{19}$-regular partition

$V = V_0 \dot\cup V_1 \dot\cup V_2 \dot\cup \dots \dot\cup V_t$ of $\mathcal{H}$ such that the partition classes satisfy

$$|V_1| = \dots = |V_t| = (2k-2\ell-1)(k-1)m$$

for some integer $m > 0$. Clearly, $|V_0| \le 2\epsilon_{19} n \le \epsilon n/3$ and $n/t \ge |V_i| \ge n/(2t)$ for all $i \in [t]$.

For the k-uniform cluster hypergraph $\mathcal{K} = \mathcal{K}(2\epsilon_{19}, \gamma/6)$ of $\mathcal{H}$ on the vertex set $[t]$ we know by Proposition 21 that all but at most $\sqrt{2\epsilon_{19}} t^{k-1} \le \epsilon_{34} t^{k-1}$ of the $(k-1)$-sets $S \in \binom{[t]}{k-1}$ satisfy

$$\deg_{\mathcal{K}}(S) \ge \left(\frac{1}{2(k-\ell)} + \frac{\gamma}{4} \right) t.$$

Thus, by Lemma 34 we find a $\mathcal{F}_{k,\ell}$-packing of $\mathcal{K}$ such that the number of uncovered vertices is at most $(5\epsilon_{34})^{1/(k-1)} t \le \epsilon t/3$.

Let $\mathcal{F}$ be an arbitrary copy of $\mathcal{F}_{k,\ell}$ in the cluster hypergraph $\mathcal{K}$ with the vertex set, say, $V(\mathcal{F}) = \{1, 2, \dots, (2k-2\ell)(k-1)\}$ grouped into sets $A_1, \dots, A_{2k-2\ell-1}, B$, all of the same size $k-1$. The edges of $\mathcal{F}$ are the sets $A_i \cup \{b\}$ with $i \in [2k-2\ell-1]$ and $b \in B$. We will show that the corresponding induced hypergraph $\mathcal{H}_{\mathcal{F}} = \mathcal{H}[V_1 \dot\cup V_2 \dot\cup \dots \dot\cup V_{(2k-2\ell)(k-1)}]$ can be covered by a family of at most $(2k-2\ell-1)(k-1)p_{39}$ pairwise disjoint ℓ-paths which leave at most

$$(2k-2\ell-1)(k-1)\beta m \tag{3.14}$$

vertices of $\mathcal{H}_{\mathcal{F}}$ uncovered. This would imply that the union of these families for the $\mathcal{F}_{k,\ell}$-packing contains at most $tp_{39} \le p$ pairwise disjoint ℓ-paths and the number of vertices in $\mathcal{H}$ not covered by these ℓ-paths is at most

$$|V_0| + (\epsilon t/3) \times n/t + t\beta m \le \epsilon n,$$

as stated in the lemma.

To find a family of ℓ-paths satisfying (3.14) let $i \in [2k-2\ell-1]$ and by suppressing the dependence on i let $a_1, \dots, a_{k-1}$ be the elements of A_i. For each $i \in [2k-2\ell-1]$ and each $a \in A_i$ we subdivide V_a into $(k-1)$ pairwise disjoint sets $U_a^1, \dots, U_a^{k-1}$, each having

$$\frac{|V_a|}{k-1} = (2k-2\ell-1)m$$

vertices and, subsequently we group them into tuples $(U_{a_1}^r, \dots, U_{a_{k-1}}^r)$ with $r \in [k-1]$. Moreover, for all $b \in B$ we subdivide V_b into $(2k-2\ell-1)$ pairwise disjoint sets, each of size

$$\frac{|V_b|}{(2k-2\ell-1)} = (k-1)m.$$

Thus, we obtain $(2k-2\ell-1)(k-1)$ such sets and there is a bijection between those sets and the $(k-1)$-tuples $(U_{a_1}^r, \dots, U_{a_{k-1}}^r)$. We fix such a bijection (arbitrarily) and denote the preimage of $(U_{a_1}^r, \dots, U_{a_{k-1}}^r)$ by W_i^r (recall that we suppressed the dependence of $a_1, \dots, a_{k-1}$ on i).

For each $i \in [2k-2\ell-1]$ and each $b \in B$ the set $A_i \cup \{b\}$ forms an edge in $\mathcal{K}$, i.e.

the tuple $(V_{a_1}, \ldots, V_{a_{k-1}}, V_b)$ is $(2\epsilon_{19}, \gamma/6)$-regular. Due to Fact 18 and $2\epsilon_{19} \leq \epsilon_{39}/2k$ we derive that the k-tuples $(U^r_{a_1}, \ldots, U^r_{a_{k-1}}, W^r_i)$ are all $(\epsilon_{39}, \gamma/6)$-regular. Hence, for each $i \in [2k-2\ell-1]$ and each $r \in [k-1]$ we can apply Lemma 39 to $(U^r_{a_1}, \ldots U^r_{a_{k-1}}, W^r_i)$ to obtain a family of at most p_{39} pairwise disjoint ℓ-paths which cover all but at most βm vertices of $(U^r_{a_1}, \ldots, U^r_{a_{k-1}}, W^r_i)$. Since there are exactly $(2k-2\ell-1)(k-1)$ such k-tuples we obtain at most $(2k-2\ell-1)(k-1)p_{39}$ paths in total and the number of vertices in $H_{\mathcal{F}}$ not covered by those paths is at most $(2k-2\ell-1)(k-1)\beta m$, as stated in (3.14).

□

4. Perfect and Nearly Perfect Matchings

In this chapter, we want to prove the results about perfect and nearly perfect matchings. First, we want to see that Theorem 5 is essentially best possible and Theorem 7 is approximately tight for the range $r \geq k/2$. This follows from well known constructions from [80] and [97].

Proposition 40. *For all $k > r > 0$ there is a constant C such that for all $n \in k\mathbb{N}$ there are k-uniform hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2$ on n vertices with minimum r-degrees*

$$\delta_r(\mathcal{H}_1) = \binom{n-r}{k-r} - \binom{\frac{(k-1)n}{k} - r + 1}{k-r} = \left(1 - \left(\frac{k-1}{k}\right)^{k-r} - o(1)\right)\binom{n}{k-r}$$

$$\delta_r(\mathcal{H}_2) = \frac{1}{2}\binom{n}{k-r} + Cn^{k-r-1}$$

which do not contain a perfect matching.

Proof. To obtain $\mathcal{H}_1$ we split the vertex set into two sets A and B of size $|A| = \frac{n}{k} - 1$ and $|B| = \frac{(k-1)}{k}n + 1$ and take as edges of $\mathcal{H}_1$ all those k-tuples intersecting A in at least one vertex. Since every edge of a matching covers at least one vertex in A and $|A| = \frac{n}{k} - 1$ there cannot exist a perfect matching. Moreover, it is easily seen that $\delta_r(\mathcal{H}_1) = \binom{n-r}{k-r} - \binom{(k-1)n/k-r+1}{k-r}$. For $k = 3$ and $r = 1$ this is essentially the bound from Theorem 5.

For the second hypergraph $\mathcal{H}_2$ we split the vertex set into sets A and B such that $|A|$ is the maximal odd integer which does not exceed $n/2$. Further we take all edges intersecting A in a even number of vertices. Then, due to parity, $\mathcal{H}_2$ does not contain a perfect matching and the minimum r-degree is $\frac{1}{2}\binom{n}{k-1} + Cn^{k-r-1}$ for some suitably chosen C. $\square$

In Section 4.1 we introduce a few auxiliary results. In particular, we prove the Absorbing Lemma (Lemma 44) which allows us to restrict ourselves to finding nearly perfect matchings rather than perfect matchings.

The Theorem 6 is obtained from a corresponding result for nearly perfect matchings in k-uniform, k-partite hypergraphs which we prove in Section 4.2. Together with the auxiliary results in Section 4.1 this immediately implies the proofs of the upper bounds for k-uniform hypergraphs, i.e., Theorem 6, and Theorem 7. Section 4.3 contains the proof of the main result, Theorem 5, and in Section 4.4 we study the interplay of δ_1 and δ_2 in view of perfect and nearly perfect matchings in 3-uniform hypergraphs.

4.1. Auxiliary Results

4.1.1. Partitioning uniform hypergraphs

In this section we show, by a simple probabilistic argument, that there exists a partition of the vertex set of a hypergraph which distributes the vertex degrees fairly (similar results appeared in [80, 89]). We start with a folklore observation.

Proposition 41. *Let $\mathcal{H}$ be a k-uniform hypergraph on n vertices. Then there is a decomposition of the edge set of $\mathcal{H}$ into kn^{k-1} pairwise edge disjoint matchings.*

Proof. Consider the auxiliary graph G on the vertex set $E(\mathcal{H})$ in which $A, B \in E(\mathcal{H})$ are connected if and only if A and B have nonempty intersection. Then the maximum degree of G is at most $k\binom{n-1}{k-1}$. Thus G has a proper colouring using $k\binom{n}{k-1}$ colours. And since the colour classes correspond to pairwise edge disjoint matchings we obtain the proposition. $\square$

Next, let $V = V_1 \dot\cup V_2 \dot\cup \ldots \dot\cup V_k$ be an equipartition of the vertex set of a k-uniform hypergraph $\mathcal{H}$, i.e., $|V_i| = |V_j|$ for all $i, j \in [k]$. For a set $T \subset V$ we say T is crossing (with respect to $V_1, \ldots, V_k$) if for all $i \in [k]$ we have $|T \cap V_i| \leq 1$. For a crossing r-set $T = \{v_1, \ldots, v_r\}$ let

$$\deg'(T) = |\{E \in \mathcal{H} \colon\ T \subset E \text{ and } E \text{ is crossing}\}|$$

denote its k-partite degree.

Lemma 42. *For all $k > r \geq 1$ there is a n_0 such that for all $n > n_0$, $n \in k\mathbb{N}$ and every k-uniform hypergraph $\mathcal{H}$ on n vertices there is an equipartition of $V(\mathcal{H}) = V_1 \dot\cup \ldots \dot\cup V_k$ satisfying*

$$\deg'(T) \geq \frac{(k-r)!}{k^{k-r}} \deg(T) - 2(k \ln n)^{1/2} n^{k-r-1/2}$$

for each crossing r-set $T \in \binom{V}{r}$.

Proof. First set $m = k - r$ and let $V = U_1 \dot\cup \ldots \dot\cup U_k$ be a random partition of V, where each vertex appears in vertex class U_j ($j = 1, \ldots, k$) independently with probability $1/k$. For a fixed r-set $T = \{v_1, \ldots, v_r\}$ let $\mathcal{L} = \mathcal{L}(T)$ denote the link hypergraph of T which consists of the vertex set $V(\mathcal{H})$ and the edge set $\mathcal{L} = \Big\{E \in \binom{V}{m} \colon\ E \cup T \in \mathcal{H}\Big\}$. Then $\mathcal{L}$ is an m-uniform hypergraph with $\deg(v_1, \ldots, v_r)$ edges. Using Proposition 41 we decompose the edge set of $\mathcal{L}$ into at most $i_0 \leq mn^{m-1}$ nonempty pairwise edge disjoint matchings which we denote by $M_1, \ldots, M_{i_0}$.

For every $i \in [i_0]$, every edge $E \in M_i$, and every index set $J \in \binom{[k]}{m}$, we say E survived (in the partition $\bigcup_{j \in J} U_j$), if $|E \cap U_j| = 1$ for all $j \in J$. Since the partition $U_1, \ldots, U_k$ was chosen randomly we have for fixed $J \in \binom{[k]}{m}$

$$\mathbb{P}[E \text{ survived}] = \frac{m!}{k^m}.$$

Thus, for $X_{i,J} = X_{i,J}(T) = |\{E \in M_i\colon\ E \text{ survived}\}|$ we have

$$\mu_{i,J} = \mu_{i,J}(T) = \mathbb{E}\left[X_{i,J}\right] = \frac{m!}{k^m}|M_i|.$$

Now call a matching M_i bad (with respect to the chosen partition $U_1, \ldots, U_k$) if there exists a set $J \in \binom{[k]}{m}$ such that

$$X_{i,J} \leq \left(1 - \left(\frac{(4k-2)\ln n)}{\mu_{i,J}}\right)^{1/2}\right)\mu_{i,J}$$

and call T a bad set (with respect to $U_1, \ldots, U_k$) if there is at least one bad $M_i = M_i(T)$. Otherwise call T a good set. For a fixed M_i the events "E survived" with $E \in M_i$ are jointly independent, hence we can apply Chernoff's inequality (Theorem 26) and we obtain

$$\mathbb{P}\left[M_i \text{ is bad}\right] \leq \tbinom{k}{m}\exp(-(2k-1)\ln n) = \tbinom{k}{m}n^{-2k+1}.$$

Summing over all matchings M_i and recalling $i_0 \leq mn^{m-1}$ and $m \leq k-1$ yields

$$\mathbb{P}\left[\text{there is at least one bad } M_i\right] \leq i_0\tbinom{k}{m}n^{-2k+1} \leq n^{-k}$$

and summing over all r-sets T we obtain

$$\mathbb{P}\left[\text{there is at least one bad } T\right] \leq n^r n^{-k} \leq n^{-1}.$$

Moreover, Chernoff's inequality yields

$$\mathbb{P}\left[\exists\, k_0 \in [k]\colon\ |U_{k_0}| > n/k + n^{1/2}(\ln n)^{1/4}/k\right] \leq k\exp(-(\ln n)^{1/2}/(3k)) = o(1)\,.$$

Thus, with positive probability there is a partition $U_1, \ldots, U_k$ such that all r-sets T are good and such that

$$|U_j| \leq n/k + n^{1/2}(\ln n)^{1/4}/k \quad \text{for every } j \in [k]\,.$$

Consequently, by redistributing at most $n^{1/2}(\ln n)^{1/4}$ vertices of the partition $U_1, \ldots, U_k$ we obtain an equipartition partition $V = V_1 \dot\cup \ldots \dot\cup V_k$ with

$$|V_j| = n/k \quad \text{and} \quad |U_j \setminus V_j| \leq n^{1/2}(\ln n)^{1/4}/k \quad \text{for every } j \in [k]\,.$$

To verify that the partition $V_1, \ldots, V_k$ satisfies the claim of the lemma note that for a crossing r-set T and the m-set $J = \{j \in [k] \colon T \cap V_j = \emptyset\}$ we have

$$\begin{aligned}
\deg'(T) &\geq \sum_{i \in [i_0]} \left(1 - \left(\frac{(4k-2)\ln n)}{\mu_{i,J}(T)}\right)^{1/2}\right) \mu_{i,J}(T) - m\frac{n^{1/2}(\ln n)^{1/4}}{k} n^{m-1} \\
&\geq \sum_{i \in [i_0]} \mu_{i,J}(T) - ((4k-2)\ln n)^{1/2} \sum_{i \in [i_0]} (\mu_{i,J}(T))^{1/2} - (\ln n)^{1/4} n^{m-1/2} \\
&= \frac{m!}{k^m} \deg(T) - ((4k-2)\ln n)^{1/2} \sum_{i \in [i_0]} (\mu_{i,J}(T))^{1/2} - (\ln n)^{1/4} n^{m-1/2} \,.
\end{aligned}$$

The Cauchy-Schwarz inequality then gives

$$\sum_{i \in [i_0]} (\mu_{i,J}(T))^{1/2} \leq \left(i_0 \sum_{i \in [i_0]} \mu_{i,J}(T)\right)^{1/2} \leq \left(mn^{m-1}\binom{n}{m}\right)^{1/2} \leq n^{m-1/2}.$$

This implies that for the partition $V_1, \ldots, V_k$ every crossing r-set T satisfies

$$\begin{aligned}
\deg'(T) &\geq \frac{m!}{k^m} \deg(T) - ((4k-2)^{1/2} + (\ln n)^{-1/4})(\ln n)^{1/2} n^{m-1/2} \\
&\geq \frac{m!}{k^m} \deg(T) - 2(k \ln n)^{1/2} n^{m-1/2} \,,
\end{aligned}$$

which proves the lemma. □

4.1.2. Absorbing Lemma

In this section we prove an *absorbing lemma*, Lemma 44. In the previous chapter, Chapter 3, we have already seen the idea of absorption in the context of finding Hamilton cycles (see Lemma 28). The absorbing lemma in the context of matchings asserts the existence of a small and powerful matching in a hypergraph with high minimum degree which, by "absorbing" vertices, creates a perfect matching provided a nearly perfect matching was found.

First, consider the following simple proposition.

Proposition 43. *Let $\mathcal{H}$ be a k-uniform hypergraph on n vertices. For all $x \in [0, 1]$ and all integers $m \leq \ell$ the following is true. If*

$$\delta_\ell(\mathcal{H}) \geq x\binom{n}{k-\ell}, \quad \textit{then} \quad \delta_m(\mathcal{H}) \geq x\binom{n}{k-m} - O(n^{k-m-1}) \,,$$

where the constant in the error term depends on k, ℓ, and m only.

Proof. Consider an arbitrary m-set $T = \{v_1, \ldots, v_m\} \in \binom{V(\mathcal{H})}{m}$. Then the condition on

$\delta_\ell(\mathcal{H})$ implies that T is contained in at least

$$\binom{k-m}{\ell-m}^{-1} \sum_{\{v_{m+1},\dots,v_\ell\}\in\binom{V\setminus T}{\ell-1}} \deg(v_1,\dots,v_\ell) \geq \binom{k-m}{\ell-m}^{-1}\binom{n-m}{\ell-m} x\binom{n}{k-\ell}$$
$$\geq x\binom{n}{k-m} - O(n^{k-m-1})$$

edges, and the proposition follows. □

Lemma 44 (Absorbing lemma)**.** *For all $\gamma > 0$ and integers $k > r \geq 1$ there is an n_0 such that for all $n > n_0$ the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph on n vertices with minimum r-degree $\delta_r(\mathcal{H}) \geq (1/2 + 2\gamma)\binom{n}{k-r}$, then there exists a matching M in $\mathcal{H}$ of size $|M| \leq \gamma^k n/k$ such that for every set $W \subset V$ of size at most $\gamma^{2k} n \geq |W| \in k\mathbb{N}$ there exists a matching covering exactly the vertices in $V(M) \cup W$.*

Proof. Given a k-uniform hypergraph H with $\delta_r(\mathcal{H}) \geq (1/2 + 2\gamma)\binom{n}{k-r}$. From Proposition 43 we know $\delta_1(\mathcal{H}) \geq \left(\frac{1}{2} + \gamma\right)\binom{n}{k-1}$ (for all large n) and it suffices to prove the lemma for $r = 1$.

Throughout the proof we assume (without loss of generality) that $1/9 \geq \gamma$ and let n_0 be chosen sufficiently large. Further set $m = k(k-1)$ and call a set $A \in \binom{V}{m}$ of size m an **absorbing** m-set for $T = \{v_1,\dots,v_k\} \in \binom{V}{k}$ if A spans a matching of size $k-1$ and $A \cup T$ spans a matching of size k, i.e., $\mathcal{H}[A]$ and $\mathcal{H}[A \cup T]$ both contain a perfect matching.

Claim 45. *For every $T = \{v_1,\dots,v_k\} \in \binom{V}{k}$ there are at least $\gamma^{k-1}\binom{n}{k-1}^k/2$ absorbing m-sets for T.*

Proof. Let $T = \{v_1,\dots,v_k\}$ be fixed. Since n_0 was chosen large enough there are at most $(k-1)\binom{n}{k-2} \leq \gamma\binom{n}{k-1}$ edges which contain v_1 and v_j for some $j \in \{2,\dots,k\}$. Due to the minimum degree of $\mathcal{H}$ there are at least $\binom{n}{k-1}/2$ edges containing v_1 but none of the vertices $v_2,\dots,v_k$. We fix one such edge $\{v_1,u_2,\dots,u_k\}$ and set $U_1 = \{u_2,\dots,u_k\}$. For each $i = 2,3,\dots,k$ and each pair u_i, v_i suppose we succeed to choose a set U_i such that U_i is disjoint to $W_{i-1} = \bigcup_{j\in[i-1]} U_j \cup T$ and both $U_i \cup \{u_i\}$ and $U_i \cup \{v_i\}$ are edges in $\mathcal{H}$. Then, for a fixed $i = 2,\dots,k$ we call such a choice U_i good, motivated by $W_k = \bigcup_{i\in[k]} U_i$ being an absorbing m-set for T.

Note that in each step $2 \leq i \leq k$ there are $k + (i-1)(k-1) \leq k^2$ vertices in W_{i-1}, thus the number of edges intersecting u_i (or w_i respectively) and at least one other vertex in W_{i-1} is at most $k^2\binom{n}{k-2}$. So the restriction on the minimum degree implies that for each $i \in \{2,\dots,k\}$ there are at least $2\gamma\binom{n}{k-1} - 2k^2\binom{n}{k-2} \geq \gamma\binom{n}{k-1}$ choices for U_i and in total we obtain $\gamma^{k-1}\binom{n}{k-1}^k/2$ absorbing m-sets for T. □

Continuing the proof of the Lemma 44, let $\mathcal{L}(T)$ denote the family of all those m-sets absorbing T. From Claim 45 we know $|\mathcal{L}(T)| \geq \gamma^{k-1}\binom{n}{k-1}^k/2$.

Now, choose a family $\mathcal{F}$ of m-sets by selecting each of the $\binom{n}{m}$ possible m-sets independently with probability

$$p = \gamma^k n/\Delta \quad \text{with} \quad \Delta = \binom{n}{k-1}^k \geq n\binom{n}{m-1} \geq m\binom{n}{m}. \tag{4.1}$$

Then, by Chernoff's bound (Theorem 26), with probability $1-o(1)$, as $n \to \infty$ the family $\mathcal{F}$ fulfills the following properties:

$$|\mathcal{F}| \leq \gamma^k n/m \tag{4.2}$$

and

$$|\mathcal{L}(T) \cap \mathcal{F}| \geq \gamma^{2k-1} n/3 \quad \forall T \in \binom{V}{k}. \tag{4.3}$$

Furthermore, using (4.1) we can bound the expected number of intersecting m-sets by

$$\binom{n}{m} \times m \times \binom{n}{m-1} \times p^2 \leq \gamma^{2k} n.$$

thus, using Markov's bound, we derive that with probability at least 1/2

$$\mathcal{F} \text{ contains at most } 2\gamma^{2k} n \text{ intersecting pairs.} \tag{4.4}$$

Hence, with positive probability the family $\mathcal{F}$ has all the properties stated in (4.2), (4.3) and (4.4). By deleting all the intersecting and non-absorbing m-sets in such a family $\mathcal{F}$ we get a subfamily $\mathcal{F}'$ consisting of pairwise disjoint absorbing m-sets which, due to $\gamma \leq 1/9$, satisfies

$$|\mathcal{L}(T) \cap \mathcal{F}'| \geq \gamma^{2k-1} n/3 - 2\gamma^{2k} n \geq \gamma^{2k} n \quad \forall T \in \binom{V}{m}.$$

So, since $\mathcal{F}'$ consists of pairwise disjoint absorbing m-sets, $\mathcal{H}[V(\mathcal{F}')]$ contains a perfect matching of size at most $\gamma^k n/k$. Further, for any subset $W \subset V$ of size $\gamma^{2k} n \geq |W| \in k\mathbb{N}$ we can partition W into at most $\gamma^{2k} n/k$ sets of size k and successively absorb them using a different absorbing m-set each time. Thus there exists a matching covering exactly the vertices in $V(\mathcal{F}') \cup W$. $\square$

As a consequence we obtain the following.

Corollary 46. *For all $\gamma > 0$ and $k > r \geq 1$ there is an n_0 such that for all $n_0 \leq n \in k\mathbb{N}$ the following holds: If $\mathcal{H}$ is a k-uniform hypergraph on n vertices with minimum r-degree $\delta_r(\mathcal{H}) \geq (1/2 + 2\gamma)\binom{n}{k-r}$ and for any set $U \subset V$ of size $|U| \leq \gamma^k n$ the remaining hypergraph $\mathcal{H}[V \setminus U]$ has a matching covering all but at most $\gamma^{2k} n$ vertices. Then $\mathcal{H}$ has a perfect matching.*

Proof. Let γ, k, and r be given. We applyLemma 44 to obtain n_0. Let $\mathcal{H}$ be a k-uniform hypergraph on $n \geq n_0$ vertices with minimum r-degree $\delta_r(\mathcal{H}) \geq (1/2 + 2\gamma)\binom{n}{k-r}$. Then using Lemma 44 we can remove a matching M of size $\gamma^k n/k$ from $\mathcal{H}$. According to the assumption, the remaining hypergraph $\mathcal{H}[V \setminus V(M)]$ contains a matching M' such that, W, the set of the uncovered vertices has size at most $\gamma^{2k} n \geq |W| \in k\mathbb{N}$. But due to Lemma 44 there is a matching covering exactly those vertices in $V(M) \cup W$, which together with M' forms a perfect matching of $\mathcal{H}$. $\square$

4.2. General upper bounds for k-uniform hypergraphs

In this section we prove Theorem 6 and Theorem 7. For this we verify general upper bounds on the minimum r-degree, which guarantee the existence of a perfect matching and nearly perfect matching in a k-uniform hypergraphs $\mathcal{H}$. This will be derived from a corresponding result on nearly perfect matchings for k-uniform, k-partite hypergraphs. Here the minimum r-degree $\delta_r(\mathcal{H})$ of a k-uniform, k-partite hypergraph with vertex partition $V_0 \dot\cup \ldots \dot\cup V_{k-1}$ is $\min \deg(v_{i_1}, \ldots, v_{i_r})$, where the minimum runs over all index sets $\{i_1, \ldots, i_r\} \in \binom{\{0,\ldots,k-1\}}{r}$ and all r-sets of vertices $v_{i_j} \in V_{i_j}$ for $j = 1, \ldots, r$.

Theorem 47. *Let $\mathcal{H}$ be a k-uniform, k-partite hypergraph on the partition classes $V_0, \ldots V_{k-1}$ each of size $|V_i| = n$ and suppose the minimum r-degree o f $\mathcal{H}$ is*

$$\delta_r(\mathcal{H}) > \frac{k-r}{k} n^{k-r} + kn^{k-r-1}.$$

Then $\mathcal{H}$ contains a matching covering all but $(r-1)k$ vertices. In particular, for $r = 1$ the matching is perfect.

Let $\mathcal{H}$ be a k-uniform, k-partite hypergraph on the partition classes $V_0, \ldots V_{k-1}$ and let M be a matching in $\mathcal{H}$. Let $v_i(E) = E \cap V_i$ for an edge $E \in \mathcal{H}$ and for notational convenience all additions are in $\mathbb{Z} \setminus k\mathbb{Z}$. Let $T_i = (v_i, v_{i+1}, \ldots, v_{i+r-1})$ with $i \in \mathbb{Z} \setminus k\mathbb{Z}$ and $v_j \in V_j$ for all $j \in \{i, \ldots, i+r-1\}$ and let $\mathcal{E} = (E_0, E_1, \ldots, E_{k-r-1}) \in [M]_{k-r}$ be a $(k-r)$-tuple of matching edges. We say T_i is **adjacent** to $\mathcal{E}$ (and vice versa) if

$$\{v_i, \ldots, v_{i+r-1}, v_{i+r}(E_0), \ldots v_{i+k-1}(E_{k-r-1})\} \in \mathcal{H}.$$

The set

$$N(T_i, (E_0, \ldots, E_{k-r-1})) = \{v_{i+r}(E_0), \ldots v_{i+k-1}(E_{k-r-1})\}$$

is called the **neighbour** of T **with respect to** $\mathcal{E}$ and by $\deg(T_i, [M]_{k-r})$ we denote the number of $(k-r)$-tuples $\mathcal{E} \in [M]_{k-r}$ the tuple T_i is adjacent to.

Proof of Theorem 6. For the proof keep in mind that all additions are considered in $\mathbb{N} \setminus k\mathbb{N}$. Take M to be a largest matching in $\mathcal{H}$. By adding arbitrary k-tuples if necessary, we may assume without loss of generality that $|M| = n - r$. Then there are rk unmatched vertices which we divide into k pairwise disjoint sets $T_0, \ldots, T_{k-1}$ with $T_i = \{v_i, v_{i+1} \ldots, v_{i+r-1}\}$ where $v_j \in V_j$.

For an arbitrary edge $E \in \mathcal{H}$ we say E is M-non-crossing if there is an $F \in M$ such that $|E \cap F| \geq 2$. Note that the number of M-non-crossing edges containing a fixed T_i is at most $k|M|^{k-r-1}$. Hence, the restriction on the minimum r-degree implies

$$\deg(T_i, [M]_{k-r}) \geq \delta_r(\mathcal{H}) - kn^{k-r-1} > \frac{k-r}{k} n^{k-r}.$$

And since this is true for each $T_i, i \in \{0, \ldots, k-1\}$ the total degree is

$$\deg(T_0 \ldots T_{k-1}, [M]_{k-r}) := \sum_{i \in \{0,\ldots,k-1\}} \deg(T_i, [M]_{k-r}) > (k-r)n^{k-r}.$$

Then, by averaging, we conclude that there must be a $(k-r)$-tuple of matching edges $(E_0, \ldots, E_{k-r-1})$ which is adjacent to at least $(k-r+1)$ tuples T_i. Without loss of generality let those T_i be $T_0, \ldots, T_{k-r}$. It is immediate from the definition that $N(T_i, (E_0, \ldots, E_{k-r-1})) = \{v_{i+r}(E_0), \ldots v_{i+k-1}(E_{k-r-1})\}$, the neighbours of those T_i with respect to $(E_0, \ldots, E_{k-r})$, are pairwise disjoint. And since each pair T_i and $N(T_i, (E_0, \ldots, E_{k-r-1}))$ form an edge in $\mathcal{H}$ the $(k-r+1)$ tuples T_i and their neighbours $N(T_i, (E_0, \ldots, E_{k-r-1}))$ form a matching of size $(k-r+1)$ in $\mathcal{H}$. Replacing $E_0, \ldots, E_{k-r-1}$ by this matching we obtain a larger matching. □

Proof of Theorem 6. Let n_0 be as asserted by Lemma 42 for given k and r. Next let $\mathcal{H}$ be a k-uniform hypergraph on $n > n_0$ vertices, $n \in k\mathbb{N}$, with minimum r-degree

$$\delta_r(\mathcal{H}) \geq \frac{k-r}{k} \binom{n}{k-r} + k^{k+1}(\ln n)^{1/2} n^{k-r-1/2}.$$

According to Lemma 42 there is a partition of $V = V(\mathcal{H})$ into k partition classes $V = V_0 \dot\cup \ldots \dot\cup V_{k-1}$ such that $|V_i| = |V_j| = n/k =: m$ for all i, j and every crossing r-set T satisfies

$$\deg'(T) \geq \frac{(k-r)!}{k^{k-r}} \delta_r(\mathcal{H}) - 2(k \ln n)^{1/2} n^{k-r-1/2}.$$

Using $(m)_{k-r} \geq m^{k-r} - m^{k-r-1} \sum_{i \in [k-r]} i$ a simple calculation yields

$$\deg'(T) \geq \frac{k-r}{k} m^{k-r} + km^{k-r-1}$$

for all crossing r-sets T. By Theorem 47 this ensures a matching covering all but $(r-1)k$ vertices. □

Proof of Theorem 7. Let $\gamma > 0$ and integers $k > r > 0$ be given. Applying Corollary 46 with $\gamma_1 = \gamma/4$ and k, r we obtain n_0'. Applying Theorem 6 with the same k and r we obtain n_0''. Set $n_0 = \max\{n_0', 2n_0'', 4k^{4k}/\gamma^2\}$ and let $\mathcal{H}$ be a k-uniform hypergraph on

$k\mathbb{N} \ni n > n_0$ vertices with minimum r-degree

$$\delta_r(\mathcal{H}) \geq \left(\max\left\{\frac{1}{2}, \frac{k-r}{k}\right\} + \gamma\right)\binom{n}{k-r}.$$

We want to apply Corollary 46 and pick an arbitrary set U of size $|U| \leq \gamma_1^k n$. Then the remaining graph $\mathcal{H}_U = \mathcal{H}[V \setminus U]$ has minimum degree

$$\delta_r(\mathcal{H}_U) \geq \delta_r(\mathcal{H}) - 3\gamma_1^k n\binom{n}{k-r-1} \geq \left(\max\left\{\frac{1}{2}, \frac{k-r}{k}\right\} + \frac{\gamma}{2}\right)\binom{n}{k-r}$$

According to Theorem 6 there is a matching in $\mathcal{H}_U$ which covers all but at most $(r-1)k \leq \gamma_1^{2k} n$ vertices. Thus, by Corollary 46, $\mathcal{H}$ contains a perfect matching. $\square$

Note that according to Proposition 40 for $r \geq k/2$ the Theorem 7 is best possible up to the constant γ.

4.3. Asymptotic bound for 3-uniform hypergraphs

In this section we prove Theorem 5. The major part is devoted to proving the existence of a matching covering $(1 - o(1))n$ vertices in a 3-uniform hypergraph with sufficiently high minimum degree. Together with Corollary 46 it will immediately imply Theorem 5.

Definition 48. *Let M be a matching in a 3-uniform hypergraph $\mathcal{H}$. For a vertex $v \in V(\mathcal{H})$ we define the link graph of v on the edge set $E_1E_2 \dots E_k \in \binom{M}{k}$ to be the graph $L_v(E_1 \dots E_k)$ with vertex set $\bigcup_{i\in[k]} E_i$ and edge set*

$$\{ab\colon\ \exists i, j \in [k], i \neq j \text{ such that } a \in E_i, b \in E_j \text{ and } vab \in \mathcal{H}\}.$$

Observe that for a large matching M covering all but $o(n)$ vertices of the hypergraph $\mathcal{H}$ we have $e(L_v(M)) \approx \deg(v)$. We will study the link graphs $L_v(M)$ of the vertices $v \in V(\mathcal{H}) \setminus V(M)$ with respect to a largest matching M in $\mathcal{H}$. Our goal is to derive a contradiction by showing that either M can be enlarged or $\mathcal{H}$ must have a rigid structure, which will violate the minimum degree condition of $\mathcal{H}$.

The following statements will be useful for the analysis of the link graph.

Fact 49. *Let B be a bipartite graph on six vertices with the vertex classes $E = \{e_1, e_2, e_3\}$ and $F = \{f_1, f_2, f_3\}$. Then the following holds:*

1. *if $e(B) \geq 7$ then B contains a perfect matching,*
2. *if $e(B) = 6$ then either B contains a perfect matching or is isomorphic to B_{033} (see Figure 4.1),*
3. *if $e(B) = 5$ then either B contains a perfect matching or B is isomorphic to a graph in $\{B_{023}, B_{113}\}$ (see Figure 4.1).*

Proof. Suppose $\deg(e_1) \le \deg(e_2) \le \deg(e_3)$. Then from $e(B) \ge 7$ we infer $\deg(e_1) \ge 1$, $\deg(e_2) \ge 2$ and $\deg(e_3) \ge 3$, thus B contains a perfect matching.

For $e(B) = 5$ we consider two cases: $\deg(e_1) = 0$ and $\deg(e_1) = 1$. In the first case we have $\deg(e_2) = 2$ and $\deg(e_3) = 3$ and B is isomorphic to B_{023}. If $\deg(e_1) = 1$ then again we distinguish two cases. If $\deg(e_2) = 2$ then $\deg(e_3) = 2$ and B is either isomorphic to B_{023} or contains a perfect matching. Else $\deg(v_2) = 1$ and $\deg(v_3) = 3$ and in this case either B is isomorphic to B_{113} or contains a perfect matching.

Finally we consider $e(B) = 6$. Observe that adding one edge to B_{113} we obtain a graph with a perfect matching since one vertex class has the degree sequence $1, 2, 3$. Adding an edge to B_{023} we see that the resulting graph contains a perfect matching unless it is isomorphic to B_{033}. □

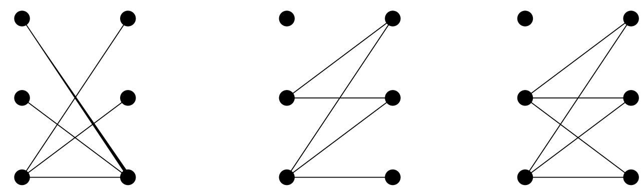

Figure 4.1.: The critical graphs: the only balanced bipartite graph(s) on six vertices and six (or five, resp.) edges without a perfect matching.

The following theorem asserts the existence of a matching covering all but $o(n)$ vertices.

Theorem 50. *For all $\gamma > 0$ there is a n_0 such that for all $n > n_0$ the following holds. Suppose $\mathcal{H}$ is a 3-uniform hypergraph on n vertices with minimum degree $\delta(\mathcal{H}) \ge (5/9 + 4\gamma)\binom{n}{2}$ then $\mathcal{H}$ contains a matching leaving strictly less than γn vertices unmatched.*

Proof. For a given $\gamma > 0$ we define $\epsilon = \gamma/150$ and by applying Theorem 24 with $\epsilon' = \min\{\gamma^2, \epsilon\}$ we obtain c and n_0'. Then we choose

$$n_0 = \max\{2^{110}/\epsilon^5, 2^{50}/c\epsilon^4, n_0'/\epsilon\}.$$

Next let M be a maximum matching of maximum size in the given hypergraph $\mathcal{H}$ and suppose $|M| = \lfloor (1-\gamma)n/3 \rfloor$. Otherwise we can simply add arbitrary 3-tuples to M to guarantee equality, since we will show that M is not a maximum matching. Let $X = V(\mathcal{H}) \setminus V(M)$ be the set of the uncovered vertices. Then from the restriction on the minimum degree we infer that the number of edges in the link graph of every vertex $v \in X$ with respect to M is

$$e(L_v(M)) \ge \deg_{\mathcal{H}}(v) - 3|M| - |X|(n - |X|) > \left(\frac{5}{9} + \gamma\right)\binom{n}{2}. \tag{4.5}$$

To derive a contradiction to (4.5) it is sufficient to show that there is a vertex $v \in X$ such that the pairs $EF \in \binom{M}{2}$ satisfying $e(L_v(EF)) \geq 6$ contribute at most $30\epsilon n^2$ edges to $L_v(M)$ in total, since then we would obtain

$$e(L_v(M)) \leq 5\binom{|M|}{2} + 30\epsilon n^2 < \left(\frac{5}{9} + \gamma\right)\binom{n}{2}. \tag{4.6}$$

We first prove the following fact.

Fact 51. *There are no $v_1v_2v_3 \in \binom{X}{3}$ and $EF \in \binom{M}{2}$ such that*

- $L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF)$ *and*
- $L_{v_1}(EF)$ *contains a perfect matching,*

Proof. Let $E = \{a, u, x\}$, $F = \{b, w, y\}$ and let the perfect matching in $L_{v_1}(EF)$ consist of the edges ab, uw and xy. Since these edges belong to the link graph of all $v_i, 1 \leq i \leq 3$, we have that $v_1ab, v_2uw, v_3xy \in \mathcal{H}$. Thus, one can replace E and F by these three edges to obtain a larger matching with contradiction to M being the maximum matching. □

Fact 52. *Let $Y_1 \subset X$ consist of those vertices $v \in X$ for which there are at least ϵn^2 pairs $EF \in \binom{M}{2}$ such that $L_v(EF)$ contains a perfect matching. Then $|Y_1| \leq \epsilon n$.*

Proof. Consider the auxiliary bipartite graph G_1 with vertex classes Y_1 and $\binom{M}{2}$ and $\{v, EF\}$ being an edge if and only if $L_v(EF)$ contains a perfect matching. Then G_1 has at least $|Y_1|\epsilon n^2$ edges and if $|Y_1|$ exceeds ϵn, by averaging, there is a pair $EF \in \binom{M}{2}$ such that $\deg_{G_1}(EF) \geq \epsilon^2 n$. Since the number of bipartite graphs on six vertices having a perfect matching is at most 2^9 we conclude from the choice of n_0 that there are $\epsilon^2 n/2^9 \geq 3$ vertices $v_1, v_2, v_3 \in Y_1$ such that $L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF)$ and $L_{v_1}(EF)$ containing a perfect matching. This yields a contradiction to Fact 51. □

Now remove Y_1 from X to obtain the set $X_1 \subset X$ of size $|X_1| \geq \gamma n/2$. Note that from Fact 49 each vertex $v \in X_1$ satisfies the following: for all but ϵn^2 pairs $EF \in \binom{M}{2}$ the link graph $L_v(EF)$ either contains at most four edges or is isomorphic to a graph in $\{B_{113}, B_{023}, B_{033}\}$.

Next we introduce some further notations. For a vertex $v \in X$ let

- $\mathcal{A}(v) = \{EF \in \binom{M}{2} \colon L_v(EF) \simeq B_{113}\}$,
- $R(v) = \{E \in M \colon$ there are ϵn elements $F \in M$ with $EF \in \mathcal{A}(v)\}$.
- $\mathcal{B}(v) = \{EF \in \binom{M}{2} \colon L_v(EF) \simeq B \in \{B_{023}, B_{033}\}\}$.

The remaining part of the proof is now devoted to showing

$$|\mathcal{B}(v)| \leq 2\epsilon n^2 \tag{4.7}$$

for some vertex $v \in X_1$. This with Fact 52 would imply

$$\begin{aligned} e(L_v(M)) &\leq 5|\mathcal{A}(v)| + 6|\mathcal{B}(v)| + 9\epsilon n^2 + 4\left(\binom{|M|}{2} - |\mathcal{A}(v)| - |\mathcal{B}(v)|\right) \\ &\leq 5\binom{|M|}{2} + 21\epsilon n^2 \end{aligned}$$

thus (4.6) follows, and by contradiction, we obtain the theorem.

To this end we first argue that there are only few pairs in $\mathcal{B}(v)$ with both elements located in $R(v)$.

Fact 53. *There are no $v_1 \dots v_5 \in \binom{X_1}{5}$ and $(E, F, G, H) \in (M)_4$ such that*

1. $L_{v_i}(EFGH) = L_{v_j}(EFGH)$ *for all* $i, j \in [5]$,
2. $\{E, F\}, \{G, H\} \in \mathcal{A}(v_1)$, *and* $\{F, G\} \in \mathcal{B}(v_1)$.

Proof. It is sufficient to show the existence of a matching of size five in $L_{v_i}(EFGH)$. With the five vertices $v_1 \dots v_5$ this yields a matching of size five in $\mathcal{H}$ and using this as replacement of $EFGH$ yields a contradiction to the maximality of M.

To this end note first that since $L_{v_1}(EF) \simeq B_{113}$ there is a vertex of degree three in each E and F which we denote by $e_1 \in E$ and $f_1 \in F$. The same holds for G and H and we denote the respective vertices by $g_1 \in G$ and $h_1 \in H$. Note that for a graph $B \in \{B_{023}, B_{033}\}$, B contains two vertices of degree two in each partition class. Consequently, since $L_{v_i}(FG) \simeq B \in \{B_{023}, B_{033}\}$ there is a vertex $f_2 \in F, f_2 \neq f_1$ which has two neighbours in G. Thus we can pick the edge f_2g_2 in $L_{v_1}(FG)$ such that $g_2 \neq g_1$. In the graph $L_{v_1}(EF)$ (and $L_{v_1}(GH)$, resp.), by using the vertices f_1, e_1 (and g_1, h_1, resp.), we now find a matching of size two which does not cover the vertex f_2 and g_2. This together yields a matching of size five in $L_{v_i}(EFGH)$. □

Fact 54. *Let $Y_2 \subset X_1$ consist of those vertices $v \in X_1$ such that there are at least ϵn^2 pairs $FG \in \binom{R(v)}{2}$ with $FG \in \mathcal{B}(v)$. Then $|Y_2| \leq \epsilon n$.*

Proof. Consider the auxiliary bipartite graph G_2 with vertex classes Y_2 and $(M)_4$ with $\{v, (E, F, G, H)\}$ being an edge if and only if $EF, GH \in \mathcal{A}(v)$ and $FG \in \mathcal{B}(v)$. Note that for each pair $FG \in \binom{R(v)}{2}$ with $FG \in \mathcal{B}(v)$, by definition of $R(v)$ we have at least $\epsilon n(\epsilon n - 1) > (\epsilon n)^2/2$ pairs $(E, H) \in (M)_2$ such that $\{v, (E, F, G, H)\} \in E(G_2)$. Hence, v has at least $\epsilon n^2(\epsilon n)^2/2$ neighbours and G_2 contains at least $|Y_2|\epsilon^3 n^4/2$ edges.

Suppose $|Y_2| > \epsilon n$ then, by averaging, there is a $EFGH \in (M)_4$ which has at least $\epsilon^4 n$ neighbours in G_2. Since the number of graphs on twelve vertices does not exceed 2^{66} from the choices of n_0 we obtain $\epsilon^4 n/2^{66} \geq 5$ vertices $v_1 \dots v_5 \in \binom{Y_1}{5}$ such that $L_{v_i}(EFGH) = L_{v_j}(EFGH)$ for all $i, j \in [5]$. This contradicts Fact 53. □

Next let $X_2 = X_1 \setminus Y_2$ and $S(v) = M \setminus R(v)$ for $v \in X_2$. Note that $|S(v)| > \epsilon n$ otherwise from the previous fact we have at most

$$\binom{|S(v)|}{2} + |R(v)||S(v)| + \epsilon n^2 \leq 2\epsilon n^2 \tag{4.8}$$

pairs in $\mathcal{B}(v)$ which by (4.7) yields the theorem. Now we argue that there are only few pairs of $\mathcal{B}(v)$ containing one element from $R(v)$ and the other from $S(v)$.

Fact 55. *There are no $v_1 \ldots v_6 \in \binom{X_2}{6}$ and $(E, F, G, H, I) \in (M)_5$ such that*

1. $L_{v_i}(EFGHI) = L_{v_j}(EFGHI)$ *for all* $i, j \in [5]$,
2. $\{E, F\}, \{H, I\} \in \mathcal{A}(v_1)$ *and* $\{F, G\}, \{G, H\} \in \mathcal{B}(v_1)$.

Proof. Again it is sufficient to prove that $L_{v_1}(EFGHI)$ contains a matching of size six. To this end first denote the vertices with degree three in $L_{v_1}(EF)$ by $e_1 \in E$, $f_1 \in F$ (and in $L_{v_1}(HI)$ by $h_1 \in H, i_1 \in I$, resp.). Since $FG \in \mathcal{B}(v_1)$ there are two vertices in G having two neighbours in F. The same holds for $GH \in \mathcal{B}(v_1)$. Thus there are $g_1, g_2 \in G, g_1 \neq g_2$ such that g_1 has two neighbours in F and g_2 has two neighbours in H. Using them we can pick two matching edges in $L_{v_1}(FGH)$ which avoid f_1 and h_1. Now the vertices e_1, f_1 (and h_1, i_1, resp.) can be extended to a matching of size two in $L_{v_1}(EF)$ (and $L_{v_1}(HI)$, resp.) which leaves the chosen neighbours of g_1 (and g_2, resp.) uncovered. Together this yields a matching of size six. □

Fact 56. *Let $Y_3 \subset X_2$ consist of all those vertices $v \in X_2$ such that there are at least ϵn^2 pairs $(E, F) \in R(v) \times S(v)$ which satisfy $EF \in \mathcal{B}(v)$. Then $|Y_3| \leq \epsilon n$.*

Proof. For a vertex $v \in Y_3$ and a $G \in S(v)$ let x_G denote the number of those $F \in R(v)$ such that $FG \in \mathcal{B}(v)$. Then there are $x_G(x_G - 1)$ choices $(F, H) \in (R(v))_2$ such that $FG, HG \in \mathcal{B}(v)$. And since $F, H \in R(v)$ we have at least $\epsilon n(\epsilon n - 1)$ choices $(E, I) \in (M)_2$ such that $EF, HI \in \mathcal{A}(v)$. Thus G gives rise to at least $x_G^2(\epsilon n)^2/2$ sets $(E, F, H, I) \in (M)_4$ satisfying $EF, HI \in \mathcal{A}(v)$ and $FG, GH \in \mathcal{B}(v)$. Recall that $s = |S(v)| > \epsilon n$ according to (4.8) and that $\sum_{G \in S(v)} x_G \geq \epsilon n^2$. From Jensen's inequality (Theorem 22) and $s < n/3$ we obtain:

$$\frac{(\epsilon n)^2}{2} \sum_{G \in S(v)} x_G^2 \geq \frac{(\epsilon n)^2}{2} s \left(\sum \frac{1}{s} x_G \right)^2 \geq \epsilon^4 n^5. \tag{4.9}$$

Such a vertex $v \in Y_3$ gives rise to at least $\epsilon^4 n^5$ ordered tuples $(E, F, G, H, I) \in (M)_5$ which satisfies $EF, HI \in \mathcal{A}(v)$ and $FG, GH \in \mathcal{B}(v)$. We consider the auxiliary bipartite graph G_3 with vertex classes Y_3 and $(M)_5$ and $\{v, (E, F, G, H, I)\}$ being an edge if and only if (E, F, G, H, I) satisfies $EF, HI \in \mathcal{A}(v)$ and $FG, GH \in \mathcal{B}(v)$. If $|Y_3|$ exceeds ϵn then G_3 contains at least $\epsilon^5 n^6$ edges. Then by averaging and the choice of n_0 we find $v_1 \ldots v_6$ which with $EFGHI$ meet the conditions in Fact 55. This, however, yields a contradiction. □

Let $X_3 = X_2 \setminus Y_3$ and note that $|X_3| \geq \gamma n/4$. Now before deriving the contradiction, we show that the density of $\mathcal{B}(v)$ in $S(v)$ is at most $1/2 + \epsilon$.

Fact 57. *There are no $v_1 \dots v_4$ and $EFG \in \binom{M}{3}$ such that*

1. $L_{v_1}(EFG) = L_{v_2}(EFG) = L_{v_3}(EFG)$,
2. $EF, FG, GE \in \mathcal{B}(v_1)$.

Proof. Similar to the previous arguments we are looking for a matching of size four in the graph $L_{v_1}(EFG)$. To this end let denote the isolated vertex in $L_{v_1}(EF)$ by x_1, the one in $L_{v_1}(FG)$ by x_2 and the one in $L_{v_1}(GE)$ by x_3. Then there are $1 \leq i, j \leq 3$ such that x_i and x_j belong to different edges and without loss of generality let $x_1 \in E$ and $x_2 \in F$. Since in the link graph $L_{v_1}(EF)$ the vertex x_1 is not adjacent to any vertex of F there must be a vertex $e_2 \in E$ which has degree three, hence is adjacent to x_2. Take e_2x_2 as the first matching edge. In the link graph $L_{v_1}(GE)$ there is a vertex $g_1 \in G$ of degree two. This we use to match a vertex $e_1 \neq e_2$ in E. Note that e_2 could equal x_1. Lastly in the link graph $L_{v_1}(FG)$ the remaining vertices $f_1 \neq x_2 \neq f_2$ have degree two and three, hence they can be used to create a matching of size two in $L_{v_1}(FG)$ which avoids the vertex g_1. Together this yields a matching of size four. $\square$

Fact 58. *Let $Y_4 \subset X_3$ contain all those vertices $v \in X_3$ such that there are at least $\left(\frac{1}{2} + \epsilon\right)\binom{S(v)}{2}$ pairs $EF \in \binom{S(v)}{2}$ such that $EF \in \mathcal{B}(v)$. Then $|Y_4| \leq \epsilon n$.*

Proof. Consider $\mathcal{B}(v) \cap \binom{S(v)}{2}$ as edges on the vertex set $S(v)$. Furthermore, note that $|S(v)| \geq \epsilon n \geq n_0$ and $\epsilon \geq \epsilon'$. Applying Theorem 24 we obtain at least $c(\epsilon n)^3$ triangles in $S(v)$, i.e., $EFG \in \binom{S(v)}{3}$ such that $EF, FG, GE \in \mathcal{B}(v)$.

As before consider the auxiliary bipartite graph G_4 on the partition classes Y_4 and $\binom{M}{3}$ with the edges $\{v, EFG\}$ if and only if $EFG \in \binom{S(v)}{3}$ and $EF, FG, GE \in \mathcal{B}(v)$. Suppose $|Y_4| > \epsilon n$, then, by averaging, we obtain a set $EFG \in \binom{M}{3}$ which is connected in G_4 to at least $c\epsilon^4 n$ vertices from Y_4. Since n was chosen in such a way that $c\epsilon^4 n/2^{40} > 3$ there are vertices $v_1v_2v_3 \in \binom{Y_4}{3}$ such that their link graphs agree on EFG, i.e., $L_{v_1}(EFG) = L_{v_2}(EFG) = L_{v_3}(EFG)$. But by Fact 57 this yields a contradiction. $\square$

Now pick a vertex $v \in X \setminus \bigcup_{i \in [4]} Y_i$. Due to the choice of v the following holds.

1. There are at most ϵn^2 pairs $EF \in \binom{M}{2}$ such that $e(L_v(EF)) \geq 7$ (due to Fact 52). So their contribution to $e(L_v(M))$ is at most $9\epsilon n^2$.
2. There are at most ϵn^2 pairs $EF \in \binom{R(v)}{2}$ such that $EF \in \mathcal{B}(v)$ (due to Fact 54), contributing at most $6\epsilon n^2$ edges to $L_v(M)$. Each of the remaining pairs have a contribution of at most 5.
3. There are at most ϵn^2 pairs $EF \in R(v) \times S(v)$ such that $EF \in \mathcal{B}(v)$ (due to Fact 56) - which yields a contribution of at most $6\epsilon n^2$. Note that by definition of $S(v)$ all but $\epsilon n|S(v)|$ of the remaining pairs from $R(v) \times S(v)$ contribute at most 4 edges to $L_v(M)$.

4. at most $\left(\frac{1}{2}+\epsilon\right)\binom{|S(v)|}{2}$ pairs $EF \in \binom{S(v)}{2}$ such that $EF \in \mathcal{B}(v)$ (due to Fact 58) which yields a contribution of at most $6(1/2+\epsilon)\binom{|S(v)|}{2}$. For all but at most $\epsilon n|S(v)|$ of the remaining pairs from $\binom{S(v)}{2}$ we have $e(L_v(EF)) \leq 4$.

Now let $r = |R(v)|$ and $s = |S(v)|$. Counting the edges in the link graph of v with respect to $M = R(v)\dot{\cup}S(v)$ we obtain from the (1)-(4) and from $s \leq |M| < n/3$

$$\begin{aligned} e(L_v(M)) \leq & 9\epsilon n^2 + \left[6\epsilon n^2 + 5\binom{r}{2}\right] + \left[6\epsilon n^2 + 5\epsilon ns + 4rs\right] \\ & + \left[6\left(\frac{1}{2}+\epsilon\right)\binom{s}{2} + 4\left(\frac{1}{2}-\epsilon\right)\binom{s}{2} + 5\epsilon ns\right] \\ \leq & 5\binom{r}{2} + 5\binom{s}{2} + 4rs + 30\epsilon n^2 \\ < & 5\binom{|M|}{2} + 30\epsilon n^2 < \left(\frac{5}{9}+\gamma\right)\binom{n}{2} \end{aligned}$$

with contradiction to (4.5). $\square$

As an immediate consequence we obtain Theorem 5.

Proof of Theorem 5. Let $\gamma > 0$ be given. Set $\gamma_1 = \gamma/4$ and $\gamma_2 = \gamma_1^6$. Applying Corollary 46 with $k = 3$, $\ell = 1$ and $2\gamma_1$ yields n_0' and applying Theorem 50 with γ_2 yields n_0''. We choose $n_0 = \max\{n_0', 2n_0''\}$. Now let $n > n_0$, $n \in 3\mathbb{N}$ and suppose $\mathcal{H}$ is a 3-uniform hypergraph on n vertices with $\delta(\mathcal{H}) \geq (5/9+\gamma)\binom{n}{2}$. Then, trivially, $\mathcal{H}$ has minimum degree $\delta(\mathcal{H}) \geq (1/2+2\gamma_1)\binom{n}{2}$ and we would like to apply Corollary 46. To this end note that for all subsets $U \subset V(\mathcal{H})$ of size at most $\gamma_1^3 n$ the remaining hypergraph $\mathcal{H}_U = \mathcal{H}[V \setminus U]$ still has minimum degree

$$\delta(\mathcal{H}_U) \geq \left(\frac{5}{9}+\frac{\gamma}{2}\right)\binom{n}{2} \geq \left(\frac{5}{9}+4\gamma_2\right)\binom{n'}{2}$$

where $n' = |V(\mathcal{H})| - |U|$. Thus, due to Theorem 50 there is a matching in $\mathcal{H}_U$ covering all but $\gamma_2 n' \leq \gamma_1^6 n$ vertices. So, we can to apply Corollary 46 and obtain a perfect matching in $\mathcal{H}$. $\square$

4.4. (Nearly) perfect matchings with several minimum degrees

In the sequel we are interested in the interplay between several minimum degree parameters of k-uniform hypergraphs. Our aim is to give an asymptotic characterisation of the existence of a perfect matching and a nearly perfect matching in terms of several minimum degrees. Recall that a nearly perfect matching in a hypergraph on n vertices is a matching covering all but a constant number of vertices. Here, we mainly focus on the asymptotic behaviour of k-uniform hypergraphs.

To be more precise let $k \geq 2$ be a fixed integer, $n \in k\mathbb{N}$ and $\gamma, x_1, \ldots, x_{k-1}$ be arbitrary positive reals, then we define the subset $\mathscr{H}_{k,n}(\gamma, x_1 \ldots, x_{k-1})$ of k-uniform hypergraphs $\mathcal{H}$ on n vertices to be

$$\mathscr{H}_{k,n}(\gamma, x_1 \ldots, x_{k-1}) = \left\{\mathcal{H}\colon\ \delta_i(\mathcal{H}) \geq (x_i + \gamma)\tbinom{n}{k-i} \text{ for } i = 1, 2, \ldots, k-1\right\}.$$

Due to Proposition 43 we have

$$\delta_i(\mathcal{H}) \geq x\binom{n}{k-i} \text{ implies } \delta_{i-1}(\mathcal{H}) \geq x\binom{n}{k-i-1} - O(n^{k-i-2}), \tag{4.10}$$

thus, we may assume $x_i \geq x_{i+1}$ for $i = 1, \ldots, k-2$.

We say $(x_1, \ldots, x_{k-1})$ **asymptotically forces a perfect matching** if for all $\gamma > 0$ there is an n_0 such that for all $n > n_0, n \in k\mathbb{N}$ every $\mathcal{H} \in \mathscr{H}_{k,n}(\gamma, x_1, \ldots, x_{k-1})$ contains a perfect matching. Similarly, we say $(x_1, \ldots, x_k)$ **asymptotically forces a nearly perfect matching** if there is a constant C such that for all $\gamma > 0$ there is an n_0 such that for all $n > n_0, n \in k\mathbb{N}$ every $\mathcal{H} \in \mathscr{H}_{k,n}(\gamma, x_1, \ldots, x_{k-1})$ contains a matching covering all but C vertices and there is an $\mathcal{H} \in \mathscr{H}_{k,n}(\gamma, x_1, \ldots, x_{k-1})$ which does not contain a perfect matching.

For arbitrary integers $k \geq 2$ we are interested in the functions

$$s_k : D_{k-1} \longrightarrow \{0, 1, 2\}$$

on the domain $D_{k-1} = \{(x_1, \ldots, x_{k-1}) \in [0,1]^k\colon\ x_i \geq x_2 \geq \ldots x_k\}$ which, with $\boldsymbol{x}$ denoting $(x_1, \ldots, x_{k-1})$, are defined by

$$s_k(\boldsymbol{x}) = \begin{cases} 2 & \boldsymbol{x} \text{ asymptotically forces a perfect matching} \\ 1 & \boldsymbol{x} \text{ asymptotically forces a nearly perfect matching} \\ 0 & \text{otherwise.} \end{cases}$$

First note that $s_k(x_1, \ldots, x_{k-1})$ is monotone increasing in each x_i. And for $k = 3$ our results determine $s_3(x_1, x_2)$ completely. We know $s_3(5/9, 0) = 2$ by Theorem 5, $s_3(1/2, 1/3) = 2$ by Theorem 6 combined with the Absorbing Lemma, Lemma 44. By Theorem 6 we know $s_3(1/3, 1/3) = 1$ and combined with the lower bounds and the monotonicity we know $s_3(x_1, x_2)$ for all $x_1 \geq x_2$ (see figure 4.2).

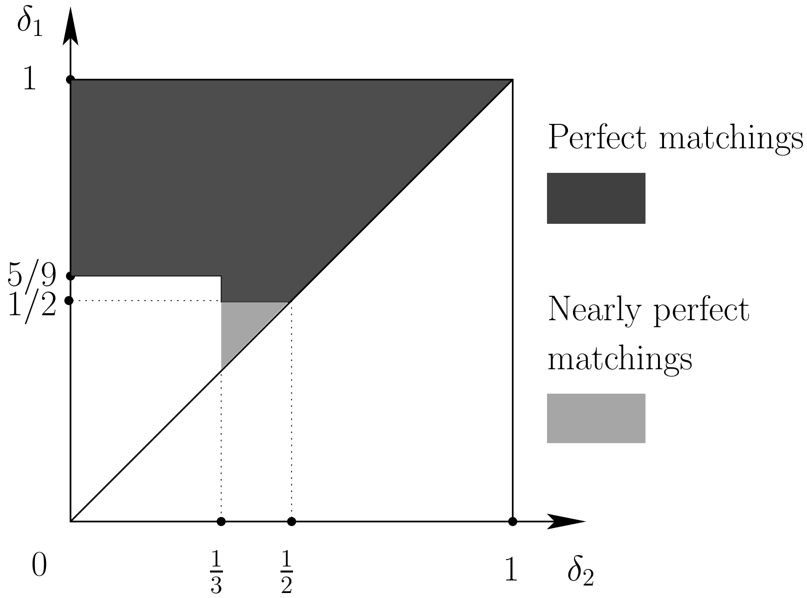

Figure 4.2.: The function $s_3(x_1, x_2)$.

Part II.

Algorithmic Regularity Lemma and Quasi-randomness

5. Prerequisites

The main purpose of this part is to prove the results described in Section 1.2.2. These results are taken from [10], a joint work with Noga Alon, Amin Coja-Oghlan, Mihyun Kang, Vojtěch Rödl and Mathias Schacht.

In Chapter 6 we prove Theorem 8 about algorithmic regularity lemma involving the new notion of regularity as introduced in (1.3). Furthermore, we show how this result can be applied to obtain a polynomial time approximation scheme for Max-Cut for graphs without too "dense spots" (Theorem 9). In Chapter 7 the proof of Theorem 10 concerning the characterisation of low discrepancy in terms of eigenvalues will be presented.

Aiming at a higher inner coherence of this scripture and the localisation of the results in their historical context we provide some background on quasi-randomness and algorithmic regularity lemma in the next section.

5.1. Historical background

The two topics, algorithmic regularity lemma and quasi-random graphs, are intimately related. As mentioned before ϵ-regularity is a "deterministic" notion which captures a very characteristic property of the binomial random bipartite graph. However, the characteristics of this object are not limited to this phenomenon of "low discrepancy" in the edge distribution and the discovery of the connections between those properties, known under the name quasi-random graphs, marks a highlight in combinatorics in the last decades. This discovery is not only interesting from the structural point of view but it also opens the door to algorithmic applications. For example, all approaches to an algorithmic version of the regularity lemma are guided by insights obtained in this field as we will briefly describe in the sequel.

Quasi-randomness for dense graphs The starting point of the theory of quasi-random graphs is the following non-partite version of regularity which is almost surely satisfied by the binomial random graph. Let $p \in (0,1)$ denote the density of the graph G, then low discrepancy with error parameter $\epsilon > 0$ is given by

$$\left| e(S) - p\binom{|S|}{2} \right| < \epsilon |V|^2 \quad \text{for all subsets } S \subset V. \tag{5.1}$$

We remark that Szemerédi's notion of regularity is a natural bipartite version of low discrepancy and all which will be mentioned in the following concerning (5.1) has a fairly straightforward bipartite analogue.

The main purpose of the study of quasi-random graphs is to provide characterisations of the property (5.1) and the systematic investigation of this problem was initiated by two papers of A. Thomason [109, 110] in 1987. Then, in the celebrated paper [22] from 1989, a corner stone in the theory of quasi-random graphs, Chung, Graham, and Wilson gave a long list of properties and proved their equivalence to (5.1). In order to state the result we introduce some notation.

Let $G = (V, E)$ be a graph on n vertices. For a graph F let $N^*_G(F)$ denote the number of labelled induced copies of F in G and let $N_G(F)$ be the number of labelled not necessarily induced copies of F in G. For a pair of vertices $x, y \in V$ we denote by $codeg(x, y)$ the number of common neighbours of x and y in G, and by $s(x, y)$ we denote the number of vertices in G with the same adjacency to x and y, i.e. either joint to both or to none. Finally, let $A = A(G)$ denote the adjacency matrix of $G = (V, E)$ which, we recall, is a $|V| \times |V|$ matrix whose entries $A(u, v)$ are 1 if $\{u, v\} \in E$ and 0 otherwise. Since A is a symmetric, $(0, 1)$-valued matrix its eigenvalues are all real numbers and we denote by λ_{max} the largest eigenvalue of A and by λ the second largest eigenvalue of A in absolute value (note that λ may equal λ_{max}).

Theorem 59 (Chung, Graham, Wilson)**.** *Let $p \in (0, 1)$ and let (G_n) be a sequence of graphs. Then the following properties are equivalent:*

$P_1(\ell)$**:** *For a fixed $\ell \geq 4$ and for all graphs L on ℓ vertices,*

$$N^*_{G_n}(L) = (1 + o(1))n^\ell p^{e(L)}(1 - p)^{\binom{\ell}{2} - e(L)}$$

$P_2(t)$**:** *Let $t \geq 4$ be an even integer and let C_t denote the cycle of length t.*

$$e(G_n) = p\binom{n}{2} + o(n^2) \quad \text{and} \quad N_{G_n}(C_t) \leq (np)^t + o(n^t).$$

P_3**:** $e(G_n) \geq p\binom{n}{2} + o(n^2)$ *and* $\lambda_{max} = (1 + o(1))np, \quad \lambda = o(n).$

P_4**:** *For each subset $U \subset V(G_n)$:* $\quad e(U) = p\binom{|U|}{2} + o(n^2).$

P_5**:** *For each subset $U \subset V(G_n)$ with $|U| = \lfloor n/2 \rfloor$:* $\quad e(U) = p\binom{|U|}{2} + o(n^2).$

P_6**:** $\sum_{x,y \in V} |s(x, y) - (p^2 + (1 - p)^2)n| = o(n^3).$

P_7**:** $\sum_{x,y \in V} |codeg(x, y) - p^2 n| = o(n^3).$

Note that the property P_4 is in fact identical to (5.1). Justified by the Theorem 59 graph sequences having any of those properties (and therefore, all) are called quasi-random and we remark that the random graph $G(n, p)$ exhibits those properties almost surely. Some of the implications have been proven earlier, e.g. [109, 5, 31] and later further quasi-random properties were discovered, e.g. [101, 102, 103, 100, 23], see also [76] for a survey.

It is remarkable that such a seemingly weak condition as P_2 is already strong enough to imply the full strength of quasi-randomness. However, to counter the impression that every reasonable property of the random graph is quasi-random note that the C_4 in property P_2 cannot be replaced by, say, odd cycles (see e.g. [22]).

Furthermore, note that among the quasi-random properties mentioned above there are quite a number (such as $P_1(\ell)$, $P_2(t)$, P_3, P_6, P_7) which are interesting from the computational point of view since they can be checked in polynomial time. In the following we will shortly describe how this can be used to prove an algorithmic version of the regularity lemma.

The algorithmic aspect of Szemerédi's regularity lemma The original proof of Szemerédi's regularity lemma [105] is based on an index increment argument. For a partition $\mathcal{P}$ of the vertex set $V = V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_t$ let the index be defined by

$$\operatorname{ind}(\mathcal{P}) = \frac{1}{t^2} \sum_{1 \le i < j \le t} d(V_i, V_j)^2$$

which is clearly bounded from above by 1. Starting with an arbitrary partition of V, Szemerédi showed the following. If a current partition $\mathcal{P}$ with t classes has more than ϵt^2 pairs which are not ϵ-regular then, by definition, for each such irregular pair (X, Y) there is a witness of ϵ-irregularity (A, B), i.e. a pair of subsets $A \subset X$ and $B \subset Y$ of size $|A| \ge \epsilon |X|$ and $|B| \ge |Y|$ which satisfies $|d(A, B) - d(X, Y)| > \epsilon$, thus certifying that (X, Y) is not ϵ-regular. By refining the current partition $\mathcal{P}$ using the witnesses of irregularity one can show that the index of the new partition $\mathcal{P}'$ has increased by a constant $f(\epsilon)$ with $f : \mathbf{R}_+ \to \mathbf{R}_+$ being a positive function, i.e. $\operatorname{ind}(\mathcal{P}') \ge \operatorname{ind}(\mathcal{P}) + f(\epsilon)$. Therefore, after $1/f(\epsilon)$ refinements the partition must satisfy the conditions stated in Szemerédi's regularity lemma.

The original proof can be turned into an algorithmic one if one can detect a witness of irregularity for a given pair (X, Y) in polynomial time. Deciding ϵ-regularity, however, is a computationally hard task.

Theorem 60. *The following problem is co-NP-complete. Let $G = (A \cup B, E)$ be a bipartite graph and let $\epsilon \le 1/2$ be given, decide whether the pair (A, B) is ϵ-regular.*

This result (first proven for $\epsilon = 1/2$ in [9] and then extended to all $1/2 > \epsilon > 0$ in [108]) is certainly discouraging but as noticed by Alon, Duke, Lefmann, Rödl and Yuster [9] it is sufficient to efficiently "approximate" ϵ-regularity. The key lemma in their approach can be formulated as follows.

Lemma 61. *There exist a function $\delta_{\mathcal{A}} : \mathbf{R}_+ \to \mathbf{R}_+$ and a polynomial time algorithm $\mathcal{A}$ which satisfy the following. Let a bipartite graph $H = (A \dot\cup B, E)$ with equal partition $|A| = |B| = n$ and an $\epsilon > 0$ be the input of the algorithm. Then it either correctly reports that (A, B) is an ϵ-regular pair or outputs a witness for the $\delta(\epsilon)$-irregularity of (A, B).*

Implicitly the lemma asserts $\delta(\epsilon) < \epsilon$ and the behaviour of the algorithm is not specified in case the pair (A, B) is ϵ-regular but not $\delta(\epsilon)$-regular. But despite this fact, the lemma is sufficient to provide a polynomial time algorithm for finding an ϵ-regular partition (cf. [9] for details).

As mentioned before there are straightforward bipartite analogues of the quasi-random properties as given in Theorem 59 which are computational trackable and which can be used for an approximation as given in the Lemma 61. In [9], e.g., the equivalence of the properties P_7 and P_4 in the bipartite form was exploited.

To explain this more in detail let H be a bipartite graph with equal colour classes $|A| = |B| = n$ and average degree d. For two vertices $b_1, b_2 \in B$ we define their neighbourhood deviation by

$$\sigma(b_1, b_2) = |codeg(b_1, b_2) - d^2/n|.$$

Moreover the deviation of a subset $Y \subset B$ is given by

$$\sigma(Y) = \frac{\sum_{y_1,y_2 \in Y} \sigma(y_1, y_2)}{|Y|^2}. \tag{5.2}$$

Note that $\sigma(B)$ is simply the normalised bipartite version of the quantity introduced in P_7. Moreover, the implication $P_4 \implies P_7$ in Theorem 59 can be restated as follows.

$P_4 \implies P_7$: For all $\epsilon > 0$ there exist a $\delta > 0$ and an n_0 such that every graph G on $n > n_0$ that satisfies

$$\left| e(U) - p\binom{|U|}{2} \right| \leq \delta n^2 \quad \text{for all} \quad U \subset V$$

also satisfies

$$\sum_{x,y \in V} |codeg(x, y) - p^2 n| \leq \epsilon n^3.$$

For the bipartite analogue the following was proven in [9].

Lemma 62. *Let $0 < \epsilon < 1/16$ and let $H = (A\dot{\cup}B, E)$ be a bipartite graph with equipartition $|A| = |B| = n$ and average degree $d > \epsilon^3 n$. Further, assume that (A, B) is not ϵ-regular. Then one of the following properties holds:*

1. *there are at least $\epsilon^4 n/8$ vertices from $b \in B$ such that $|deg(b) - d| > \epsilon^4 n$,*

2. *there exists $Y \subset B$ of size $|Y| \geq \epsilon n$ such that $\sigma(Y) \geq \epsilon^3 n/2$.*

It is easily seen that in the first case the pair (A, B) cannot be $\epsilon^4/16$-regular and that this can be checked in $O(n^2)$. In the second case, one can find the witness of irregularity as follows. Let $y_0 \in Y$ be the vertex with $|\deg(y) - d| < \epsilon^4$ which maximises $\sum_{y \in Y} \sigma(y_0, y)$ and moreover, let

$$B' = \{y \in Y \colon\ \sigma(y, y_0) > 2\epsilon^4 n\} \quad \text{and} \quad A' = N(y_0).$$

Since $\sigma(Y) \geq \epsilon^3 n/2$ it is easily seen that there is a y_0 such that

$$\sum_{y \in Y} \sigma(y_0, y) \geq \frac{3}{8}\epsilon^3 n|Y| \tag{5.3}$$

which implies $|B'| \geq \epsilon^4 n/4$. Clearly $d + \epsilon^4 n \geq |A'| \geq \epsilon^4$ and

$$e(A', B') = \sum_{b \in B'} |N(y_0) \cap N(b)| \geq \frac{|B'|d^2}{n} + 2\epsilon^4 n|B'|.$$

With $e(A, B) = dn^2$ this implies $d(A', B') - d(A, B) > \epsilon^4$ which proves that (A', B') is a witness for ϵ^4-irregularity (see [9] for more details). Since the computation of all quantities $\sigma(y, y')$ with $y, y' \in B$ can be done by squaring the adjacency matrix of H the vertex y_0 and the pair (A', B') can be found in $O(M(n))$ where $M(n) = O(n^{2.376})$ is the time needed to multiply two $n \times n$ matrices with $0, 1$ entries.

This approach has been refined by Kohayakawa, Rödl, and Thoma [63] to improve the running time of the algorithm to $O(n^2)$. The main idea in their approach is that it is not necessary to control the co-degree of all pairs $b_1, b_2 \in B$ but checking the pairs that form an edge of a linear-sized expander is indeed sufficient. Exploiting the equivalence $P_3 \iff P_4$, Frieze and Kannan [34] introduced another approach which uses singular values. Further, the use of randomisation yields a more efficient algorithm [33, 34]. We do not discuss these approaches here and refer to [34, 33, 32] as well as the survey [62] for further details.

Generalisations and related works From above we have seen how the results in the theory of quasi-random graphs and algorithmic regularity lemma are related and it is natural to ask for extensions for the case of sparse graphs, i.e. with vanishing edge density $p(n) = o(1)$, and for the even more general model of graphs with arbitrary degree distribution.

Concerning these two generalisations the picture is by far not as complete as it is for dense graphs. Most of the time the generalisations of the properties given in Theorem 59 are immediate. However, several implications which are true in the dense case fail in the sparse analogues, hence also fail for graphs with general degree distribution. For example, there exists sparse graphs with very balanced edge distribution which does not contain a single copy of a fixed graph. In case of K_k for example, the complete graph on k vertices, such a graph is easily proven to exist by taking the random graph $G(n, p)$ with $p \ll n^{-1/(k+1)}$ and subsequently deleting all copies of K_k (see also [4] for a "deterministic" example).

Quasi-random graphs with general degree distributions were first studied by Chung and Graham [21]. Among others (e.g., weighted cycles) they considered the properties Disc and Eig (as defined in (1.5) and (1.6)) and observed that Eig implies Disc. The converse, however, is not true (see Chapter 7 for an example). Regarding the step from discrepancy to eigenvalue separation, Butler [16] proved that any graph G such that for all sets $X, Y \subset V$ the bound

$$|e(X,Y) - \mathrm{vol}(X)\mathrm{vol}(Y)/\mathrm{vol}(V)| \leq \epsilon\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)} \tag{5.4}$$

holds, satisfies $\mathrm{Eig}(O(\epsilon(1-\ln\epsilon)))$. His proof builds upon the work of Bilu and Linial [11], who derived a similar result for regular graphs, and on the earlier related work of Bollobás and Nikiforov [14].

Butler's result relates to the second part of Theorem 10 as follows. The r.h.s. of (5.4) refers to the volumes of the sets X, Y, and may thus be significantly smaller than $\epsilon\mathrm{vol}(V)$. By comparison, the second part of Theorem 10 just requires that the "original" discrepancy condition $\mathrm{Disc}(\delta)$ is true, i.e., we just need to bound $|e(S) - \mathrm{vol}(S)^2/\mathrm{vol}(V)|$ in terms of the *total* volume $\mathrm{vol}(V)$. Hence, Butler shows that the "original" eigenvalue separation condition Eig follows from a stronger version of the discrepancy property. By contrast, Theorem 10 shows that the "original" discrepancy condition Disc implies a weak form of eigenvalue separation ess-Eig, thereby answering a question posed by Chung and Graham [21, 17]. Furthermore, relying on Grothendieck's inequality and duality of semidefinite programming, the proof of Theorem 10 employs quite different techniques than [11, 14, 16].

5.2. Tools, notation and basic facts

In this section we collect the tools needed for the proofs. Our approach to both problems, Theorem 8 and Theorem 10, crucially relies on properties of positive semidefinite matrices and results on semidefinite programming as well as Grothendieck's inequality whose introduction constitutes the main part of this section. The reader familiar with these concepts may skip this part and come back whenever needed. Most of the following results on positive semidefinite matrices and the normalised Laplacian matrix can be found in standard literature on linear algebra and spectral graph theory such as [52] and [20, 43]. The results on semidefinite programming and Grothendieck's constant can be found in [3, 56] and [49, 6].

First we recall the defect-form of the Cauchy-Schwarz inequality which is crucial for the index increment argument in the proof of the regularity lemma.

Lemma 63 (Defect form of Cauchy-Schwarz inequality)**.** *For all $i \in I$ let σ_i, d_i be positive real numbers satisfying $\sum_{i\in I}\sigma_i = 1$. Furthermore let $J \subset I$, $\varrho = \sum_{i\in I}\sigma_i\varrho_i$ and $\sigma_J = \sum_{j\in J}\sigma_j$. If*

$$\sum_{j\in J}\sigma_j\varrho_j = \sigma_J(\varrho+\nu)$$

then

$$\sum_{i\in I}\sigma_i\varrho_i^2 \geq \varrho^2 + \nu^2\sigma_J.$$

5.2.1. Symmetric and positive semidefinite matrices

The set of real $m \times n$ matrices can be interpreted as a vector space in $\mathbf{R}^{m \cdot n}$ which has the following natural inner product

$$\langle A, B \rangle = \operatorname{tr}(B^T A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}.$$

Here $\operatorname{tr}(\cdot)$ denotes the linear function **trace** which is simply the sum of the diagonal elements of a square matrix. The norm associated with this inner product is called the Frobenius norm.

Instead of working with arbitrary matrices we will usually deal with symmetric matrices which intrinsically carry many useful properties about their spectra. For example, all eigenvalues of a symmetric, real valued $n \times n$ matrix M are real and there is an orthonormal basis of $\mathbf{R}^n$ consisting of eigenvectors of M. Equivalently, for such a matrix M with $\operatorname{rank}(M) = k$ there is an eigenvalue decomposition consisting of an $n \times k$ matrix P with $P^T P = \boldsymbol{E}$ and diagonal real valued matrix Λ_M of rank k such that $M = P \Lambda_M P^T$. Here $\boldsymbol{E}_k$ denotes the identity matrix rank k.

For convenience we order the eigenvalues of the matrix M non-decreasingly

$$\lambda_1 [M] \leq \cdots \leq \lambda_n [M] = \lambda_{\max} [M]$$

and occasionally we will refer to the Courant-Fischer characterisations of λ_2 and $\lambda_{\max}$, which read (see [52])

$$\lambda_2 [M] = \max_{0 \neq \zeta \in \mathbf{R}^\nu} \min_{\xi \perp \zeta, \|\xi\|=1} \langle M\xi, \xi \rangle , \quad \lambda_{\max} [M] = \max_{\zeta \in \mathbf{R}^\nu, \|\zeta\|=1} \langle M\zeta, \zeta \rangle \tag{5.5}$$

Here and later, for a vector $\xi \in \mathbf{R}^V$ we let $\|\xi\|$ signify the ℓ_2-norm. Accordingly for a matrix $M \in \mathbf{R}^{V \times V}$ we let

$$\|M\| = \max_{0 \neq \xi \in \mathbf{R}^V} \frac{\|M\xi\|}{\|\xi\|}$$

denote the spectral norm which equals the spectral radius $\lambda_{\max} [M]$ for positive semidefinite matrices.

An $n \times n$ symmetric matrix M is called

- **positive semidefinite** ($M \geq 0$) if $x^T M x \geq 0$ for all $x \in \mathbf{R}^n$ and
- **positive definite** ($M > 0$) if $x^T M x > 0$ for all $x \in \mathbf{R}^n \setminus \{0\}$.

Furthermore, if M, M' are symmetric, then $M \geq M'$ (resp. $M > M'$) denotes the fact that $M - M' \geq 0$ (resp. $M - M' > 0$).

There are several characterisations of positive semidefinite matrices.

Theorem 64 (Characterisations of positive semidefinite matrices)**.** *For a symmetric $n \times n$ matrix M the following are equivalent:*

1. *M is positive semidefinite,*

2. *$\lambda_i(M) \geq 0$ for $i = 1, \ldots, n$,*
3. *there exists an $m \times n$ real-valued matrix C with $\operatorname{rank}(C) = \operatorname{rank}(M)$ such that $M = C^T C$.*

Proof. We prove the following implications: (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

Let $v \in \mathbf{R}^n$ with $||v||_2 = 1$ denote the eigenvector to the eigenvalue λ of M. By definition, we have $0 \leq v^T M v = \lambda v^T v = \lambda$, which proves the first implication. For the second implication, let $P^T M P = \Lambda$ be the eigenvalue decomposition of M and let $\Lambda^{1/2}$ be the matrix whose elements are the square roots of the elements of Λ. Then $C = \Lambda^{1/2} P$ satisfies the properties in (3). Finally, for an arbitrary vector $v \in \mathbf{R}^n$ we have $v^T M v = (Cv)^T (Cv) \geq 0$, which proves the last implication. □

Minor changes in the proof yields similar characterisations for positive definite matrices.

Theorem 65 (Characterisations of positive definite matrices)**.** *For a symmetric $n \times n$ matrix M the following are equivalent:*

1. *M is positive definite,*
2. *$\lambda_i(M) > 0$ for $i = 1, \ldots, n$,*
3. *there is an $m \times n$ real-valued matrix C with $rank(C) = n$ such that $M = C^T C$.*

The set of all positive semidefinite matrices form a cone, i.e. this set is closed under non-negative multiplication with scalars and addition. This is easily seen from the definition of positive semidefiniteness. The interior of this cone consists of the positive definite matrices and its boundary are positive semidefinite matrices with with at least one zero eigenvalue.

Further notation To make the proof more readable we use the Kronecker product which denotes the following product of a $m \times n$ matrix A with an arbitrary matrix B

$$A \otimes B = \begin{pmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \ldots & a_{mn}B \end{pmatrix}$$

Moreover, if $\xi = (\xi_v)_{v \in V}$ is a vector, then $\operatorname{diag}(\xi)$ signifies the $V \times V$ matrix with diagonal ξ and off-diagonal entries equal to 0. In particular, $\boldsymbol{E} = \operatorname{diag}(\mathbf{1})$ denotes the identity matrix (of any size). Moreover, if M is a $\nu \times \nu$ matrix, then $\operatorname{diag}(M) \in \mathbf{R}^\nu$ signifies the vector comprising the diagonal entries of M.

5.2.2. The Laplacian matrix

For the proof of Theorem 10 we use the representation of a graph called the normalised Laplacian matrix, or simply the Laplacian. This is a positive semidefinite matrix and we recapitulate some facts about this matrix to make the proof more comprehensible.

Given an n-vertex graph $G = (V, E)$ then its adjacency matrix $A = A(G)$ is an $n \times n$ matrix with the entries $A(u, v)$ equals 1 if $\{u, v\} \in E$ and 0 otherwise. The normalised Laplacian matrix $L = L(G)$ of G is an $n \times n$ matrix with the entries

$$L(v, w) = \begin{cases} 1 & \text{if } v = w \text{ and } d_v \geq 1, \\ -(d_v d_w)^{-\frac{1}{2}} & \text{if } v, w \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

where d_v denotes the degree of v.

For a given graph G let D denote the diagonal matrix with the entries $D(v, v) = d_v$ for all $v \in V$. Adopting the convention $D^{-1}(v, v) = 0$ for $d_v = 0$ we have the following relation between the Laplacian $L(G)$ and adjacency matrix $A(G)$.

$$L = D^{-1/2}(D - A)D^{-1/2} = \boldsymbol{E}_n - D^{-1/2}AD^{-1/2}. \tag{5.6}$$

Let $S = S(G)$ denote a matrix whose columns are indexed by the vertices and whose rows are indexed by the edges of G such that each row corresponding to an edge $e = \{u, v\}$ has the entry $1/\sqrt{d_u}$ in the column corresponding to u and the entry $-1/\sqrt{d_v}$ in the column corresponding to v. Since the Laplacian can be written as $L(G) = S^T S$ we know by Theorem 64 that L is positive semidefinite. As it turns out the product is independent of the choice of the signs as long as one entry is positive and the other negative. Moreover, 0 is an eigenvalue of L with the corresponding eigenvector $D^{1/2}\mathbf{1}$.

The largest eigenvalue of $L(G)$ is at most $\lambda_{\max[L]} \leq 2$ which can be derived from the Courant-Fisher characterisation (5.5) (also known as Rayleigh-quotient). Indeed, using $(a - b)^2 \leq 2(a^2 + b^2)$ we obtain the following (from which one also can see that all eigenvalues are non-negative).

$$\begin{aligned} \langle L\zeta, \zeta \rangle &= \left\langle (D - A)D^{-1/2}\zeta, D^{-1/2}\zeta \right\rangle \\ &= \sum_{\{u,v\} \in E} \left(\frac{\zeta_u}{\sqrt{d_u}} - \frac{\zeta_v}{\sqrt{d_v}} \right)^2 \\ &\leq 2 \sum_{\{u,v\} \in E} \left(\frac{\zeta_u^2}{d_u} + \frac{\zeta_v^2}{d_v} \right). \end{aligned}$$

Hence, with $\lambda_{\max}[L] = \max_{\zeta \in \mathbf{R}^\nu, \|\zeta\| = 1} \langle L\zeta, \zeta \rangle$ we obtain $\lambda_{\max}[L] \leq 2$.

5.2.3. Semidefinite programming and duality

Semidefinite programs Semidefinite programming is linear programming over the cone of semidefinite matrices. In comparison to standard linear programming the cone of the non-negative orthants $x \in \mathbf{R}^n \cup \{0\}$ is replaced by the cone of positive semidefinite matrices $X \geq 0$.

Let $C, A_1, \ldots, A_m$ be symmetric $n \times n$ matrices and $b_1, \ldots, b_m \in \mathbf{R}^n$. The standard formulation of a semidefinite program is given by the following (see e.g. [56]).

$$\begin{aligned} \mathrm{SDP} = \min\ & \langle C, X\rangle \\ \text{s.t. } & \langle X, A_i\rangle = b_i \text{ for all } i = 1, \dots, m \\ & X \geq 0. \end{aligned} \tag{5.7}$$

We note that the semidefinite program might be given as a maximisation problem as well since $\max \langle C, X\rangle = -\min \langle -C, X\rangle$. Moreover, let SDP denote the following

$$\mathrm{SDP} = \inf\{\langle C, X\rangle : \ \langle X, A_i\rangle = b_i \text{ for } i = 1, ..., m, X \geq 0\} \in \mathbf{R} \cup \{\infty\}. \tag{5.8}$$

The dual program of (5.7) is given by

$$\begin{aligned} \mathrm{DSDP} = \max\ & \langle b, y\rangle \\ \text{s.t. } & \sum_{i=1}^{m} y_i A_i + Z = C \\ & y \in \mathbf{R}^m, Z \geq 0. \end{aligned} \tag{5.9}$$

Besides the formulation as a maximising problem let DSDP denote the following number

$$\mathrm{DSDP} = \sup\left\{\langle b, y\rangle : \ \sum_{i=1}^{m} y_i A_i \leq C, y \in \mathbf{R}^m\right\} \in \mathbf{R} \cup \{-\infty\}. \tag{5.10}$$

We refer to [56] for an explanation why the dual program is indeed a semidefinite program.

Using e.g. the ellipsoid method [50] the semidefinite programs SDP and DSDP can be solved in polynomial time (within arbitrary precision) under certain assumptions. We do not state a general result on the polynomial time solvability of semidefinite programs here, since we shall encounter only well-studied and well-behaved examples. Further information can be found in Alizadeh [3] and Helmberg [56].

Duality theory The weak duality theorem for semidefinite programs states that the objective value of any dual feasible solution cannot exceed the objective value of any primal feasible solution but unlike linear programming the optimal values of the primal and the dual program may not coincide (see [56] for an example). However, in case SDP or DSDP is **strictly feasible** the inequality DSDP $\leq$ SDP is in fact an equality and this fact is known as the strong duality theorem (cf. [56]). Here, we say that the SDP is strictly feasible if there exists a feasible solution X which is positive definite. The DSDP is called strictly feasible if there exists a feasible solution (y, Z) with positive definite Z.

Theorem 66. *If either* SDP *or* DSDP *is strictly feasible, then* SDP = DSDP*. Furthermore, if* SDP *is strictly feasible, then the infimum* (5.8) *is attained, and if* DSDP *is strictly feasible, then the supremum* (5.10) *is attained.*

5.2.4. The cut-norm and Grothendieck's inequality

Now we explain how semidefinite programs will be used in our context. To this end let $M = (m_{ij})_{i,j \in \mathcal{I}}$ be a matrix. The **cut-norm** of M is

$$\|M\|_{\text{cut}} = \max_{I,J \subset \mathcal{I}} \left| \sum_{(i,j) \in I \times J} m_{ij} \right| .$$

In addition, consider the following optimisation problem:

$$\begin{aligned} \mathrm{SDP}(M) = &\max \sum_{i,j \in \mathcal{I}} m_{ij} \langle x_i, y_j \rangle \\ &\text{s.t. } \forall i \in \mathcal{I} : \|x_i\| = \|y_i\| = 1,\ x_i, y_i \in \mathbf{R}^{2|\mathcal{I}|}. \end{aligned} \tag{5.11}$$

This optimisation problem is indeed a positive semidefinite program. The proof is immediate from the characterisation of positive semidefinite matrices.

Lemma 67. *For any $\nu \times \nu$ matrix M we have*

$$\begin{aligned} \mathrm{SDP}(M) = &\frac{1}{2} \max \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M, X \right\rangle \\ &s.t.\ \mathrm{diag}(X) = \mathbf{1},\ X \geq 0,\ X \in \mathbf{R}^{2\nu \times 2\nu}. \end{aligned} \tag{5.12}$$

Proof. Assume that $x_1, \ldots, x_{2\nu} \in \mathbf{R}^{2\nu}$ is a family of unit vectors satisfying

$$\mathrm{SDP}(M) = \sum_{i,j=1}^{\nu} m_{ij} \langle x_i, x_{j+\nu} \rangle .$$

Then we obtain a positive semidefinite matrix $X = (x_{i,j})_{1 \leq i,j \leq 2\nu}$ via $x_{i,j} = \langle x_i, x_j \rangle$. Since $x_{i,i} = \|x_i\|^2 = 1$ for all i, this matrix satisfies $\mathrm{diag}(X) = \mathbf{1}$. Moreover,

$$\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M, X \right\rangle = 2 \sum_{i,j=1}^{\nu} m_{ij} x_{i,j+\nu} = 2 \sum_{i,j=1}^{\nu} m_{ij} \langle x_i, x_{j+\nu} \rangle . \tag{5.13}$$

Hence, the optimisation problem on the r.h.s. of (5.12) yields an upper bound on $\mathrm{SDP}(M)$.

Conversely, if $X = (x_{i,j})$ is a feasible solution to (5.12), then there exist vectors $x_1, \ldots, x_{2\nu} \in \mathbf{R}^{2\nu}$ such that $x_{i,j} = \langle x_i, x_j \rangle$, because X is positive semidefinite. Moreover, since $\mathrm{diag}(X) = \mathbf{1}$, we have $1 = x_{i,i} = \|x_i\|^2$. Thus, $x_1, \ldots, x_{2\nu}$ is a feasible solution to (5.11), and (5.13) shows that the resulting objective function values coincide. □

Grothendieck [49] established the following relation between $\mathrm{SDP}(M)$ and $\|M\|_{\text{cut}}$.

Theorem 68. *There is a constant $\theta > 1$ such that for all matrices M we have*

$$\|M\|_{\text{cut}} \leq \mathrm{SDP}(M) \leq \theta \cdot \|M\|_{\text{cut}} .$$

The best bounds on the above constant are $\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2\ln(1+\sqrt{2})}$ [49, 77]. Since by Lemma 67 $\mathrm{SDP}(M)$ can be stated as a semidefinite program, an optimal solution to $\mathrm{SDP}(M)$ can be approximated in polynomial time within any numerical precision. By applying an appropriate rounding procedure to a near-optimal solution to $\mathrm{SDP}(M)$, Alon and Naor [6] obtained the following algorithmic result.

Theorem 69. *There are a constant $\theta' > 0$ and a polynomial time algorithm* ***ApxCutNorm*** *that on input M computes two sets $I, J \subset \mathcal{I}$ such that $\theta' \cdot \|M\|_{\mathrm{cut}} \leq \left|\sum_{i \in I, j \in J} m_{ij}\right|$.*

Alon and Naor presented a randomised algorithm that guarantees an approximation ration $\theta' > 0.56$, and a deterministic one with $\theta' \geq 0.03$.

One intrinsic property of semidefinite programming, which is crucial in the proof of Theorem 10, is its close relationship to the computation of eigenvalues. Indeed, the oldest form of semidefinite programming is given in the form of eigenvalue evaluation of symmetric matrices and for our purpose we establish the following relationship.

Lemma 70. *For any symmetric $n \times n$ matrix Q we have*

$$\mathrm{SDP}(Q) = n \cdot \min_{z \in \mathbf{R}^n,\, z \perp \mathbf{1}} \lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right].$$

Proof. For a symmetric $n \times n$ matrix Q set $\mathcal{Q} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q$. Furthermore, let

$$\mathrm{DSDP}(Q) = \min \langle \mathbf{1}, y \rangle \ \text{ s.t. } \mathcal{Q} \leq \mathrm{diag}(y), \quad y \in \mathbf{R}^{2n}.$$

By Lemma 67 we can rewrite the vector program $\mathrm{SDP}(Q)$ in the standard form of a semidefinite program:

$$\mathrm{SDP}(Q) = \max \langle \mathcal{Q}, X \rangle \ \text{ s.t. } \mathrm{diag}(X) = \mathbf{1},\ X \geq 0,\ X \in \mathbf{R}^{(2n)\times(2n)}.$$

Since $\mathrm{DSDP}(Q)$ is the dual of $\mathrm{SDP}(Q)$ we immediately derive from Theorem 66 that

$$\mathrm{SDP}(Q) = \mathrm{DSDP}(Q) \tag{5.14}$$

holds.

To infer Lemma 70, we shall simplify DSDP and reformulate this semidefinite program as an eigenvalue minimisation problem. First, we show that it suffices to optimise over $y' \in \mathbf{R}^n$ rather than $y \in \mathbf{R}^{2n}$.

Lemma 71. *Let*

$$\mathrm{DSDP}'(Q) = \min 2 \langle \mathbf{1}, y' \rangle \ \textit{ s.t. } \mathcal{Q} \leq \mathrm{diag}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes y'\right),\ y' \in \mathbf{R}^n.$$

Then $\mathrm{DSDP}(Q) = \mathrm{DSDP}'(Q)$.

Proof. Since for any feasible solution y' to $\mathrm{DSDP}'(Q)$ the vector $y = \binom{1}{1} \otimes y'$ is a feasible solution to $\mathrm{DSDP}(Q)$, we conclude that $\mathrm{DSDP}(Q) \leq \mathrm{DSDP}'(Q)$.

To establish the converse inequality let $\mathcal{F}(Q) \subset \mathbf{R}^{2n}$ signify the set of all feasible solutions y to $\mathrm{DSDP}(Q)$. We shall prove that $\mathcal{F}(Q)$ is closed under the linear operator

$$\mathcal{I} : \mathbf{R}^{2n} \to \mathbf{R}^{2n}, \quad (y_1, \ldots, y_n, y_{n+1}, \ldots, y_{2n}) \mapsto (y_{n+1}, \ldots, y_{2n}, y_1, \ldots, y_n),$$

i.e., $\mathcal{I}(\mathcal{F}(Q)) \subset \mathcal{F}(Q)$; note that $\mathcal{I}$ just swaps the first and the last n entries of y. To see that this implies the assertion, consider an optimal solution $y = (y_i)_{1 \leq i \leq 2n} \in \mathcal{F}(Q)$. Then $\frac{1}{2}(y + \mathcal{I}y) \in \mathcal{F}(Q)$, because $\mathcal{F}(Q)$ is convex. Now, let $y' = (y'_i)_{1 \leq i \leq n}$ be the projection of $\frac{1}{2}(y + \mathcal{I}y)$ onto the first n coordinates. Since $\frac{1}{2}(y + \mathcal{I}y)$ is a fixed point of $\mathcal{I}$, we have $\frac{1}{2}(y + \mathcal{I}y) = \binom{1}{1} \otimes y'$. Hence, the fact that $\frac{1}{2}(y + \mathcal{I}y)$ is feasible for $\mathrm{DSDP}(Q)$ implies that y' is feasible for $\mathrm{DSDP}'(Q)$. Thus, we conclude that

$$\mathrm{DSDP}'(Q) \leq 2 \langle \mathbf{1}, y' \rangle = \langle \mathbf{1}, y \rangle = \mathrm{DSDP}(Q).$$

To show that $\mathcal{F}(Q)$ is closed under $\mathcal{I}$ consider a vector $y \in \mathcal{F}(Q)$. Since $\mathrm{diag}(y) - \mathcal{Q}$ is positive semidefinite, we have

$$\forall \eta \in \mathbf{R}^{2n} : \langle (\mathrm{diag}(y) - \mathcal{Q})\eta, \eta \rangle \geq 0. \tag{5.15}$$

The objective is to show that $\mathrm{diag}(\mathcal{I}y) - \mathcal{Q}$ is positive semidefinite, i.e.,

$$\forall \xi \in \mathbf{R}^{2n} : \langle (\mathrm{diag}(\mathcal{I}y) - \mathcal{Q})\xi, \xi \rangle \geq 0. \tag{5.16}$$

To derive (5.16) from (5.15), we decompose y into its two halfs $y = \binom{u}{v}$ $(u, v \in \mathbf{R}^n)$. Then $\mathcal{I}y = \binom{v}{u}$. Moreover, let $\xi = \binom{\alpha}{\beta} \in \mathbf{R}^{2n}$ be any vector, and set $\eta = \mathcal{I}\xi = \binom{\beta}{\alpha}$. We obtain

$$\begin{aligned}
\langle (\mathrm{diag}(\mathcal{I}y) - \mathcal{Q})\xi, \xi \rangle &= \langle \mathrm{diag}(v)\alpha, \alpha \rangle + \langle \mathrm{diag}(u)\beta, \beta \rangle - \frac{\langle Q\alpha, \beta \rangle + \langle Q\beta, \alpha \rangle}{2} \\
&= \langle (\mathrm{diag}(y) - \mathcal{Q})\eta, \eta \rangle \overset{(5.15)}{\geq} 0,
\end{aligned}$$

thereby proving (5.16). □

Back to the proof of Lemma 70 let

$$\mathrm{DSDP}''(Q) = n \cdot \min_{z \in \mathbf{R}^n,\, z \perp \mathbf{1}} \lambda_{\max} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q + \mathrm{diag} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes z \right].$$

By (5.14) and Lemma 71, it suffices to prove that $\mathrm{DSDP}'(Q) = \mathrm{DSDP}''(Q)$.

To see that $\mathrm{DSDP}''(Q) \leq \mathrm{DSDP}'(Q)$, consider an optimal solution y' to $\mathrm{DSDP}'(Q)$. We define

$$\lambda = n^{-1} \langle \mathbf{1}, y' \rangle \text{ and } z = 2(\lambda \mathbf{1} - y').$$

Then $\langle z, \mathbf{1} \rangle = 2(n\lambda - \langle \mathbf{1}, y' \rangle) = 0$, whence z is a feasible solution to $\mathrm{DSDP}''(Q)$. Furthermore, as y' is a feasible solution to $\mathrm{DSDP}'(Q)$, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q = 2\mathcal{Q} \leq 2\mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix} \otimes y' = 2\lambda \boldsymbol{E} - \mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix} \otimes z,$$

where $\boldsymbol{E}$ is the identity matrix. Hence, the matrix

$$2\lambda\boldsymbol{E} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q - \mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix} \otimes z$$

is positive semidefinite. This implies that all eigenvalues of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q + \mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix} \otimes z$$

are bounded by 2λ, i.e., $\lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q + \mathrm{diag}\binom{1}{1} \otimes z\right] \leq 2\lambda$. As a consequence,

$$\begin{aligned}\mathrm{DSDP}''(Q) &\leq n\lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q + \mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix} \otimes z\right] \\ &\leq 2n\lambda = 2\left\langle \mathbf{1}, y'\right\rangle = \mathrm{DSDP}'(Q).\end{aligned}$$

Conversely, consider an optimal solution z to $\mathrm{DSDP}''(Q)$. Set

$$\mu = \lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q + \mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix} \otimes z\right] = n^{-1}\mathrm{DSDP}''(Q)$$

and

$$y' = \frac{1}{2}(\mu\mathbf{1} - z).$$

Since all eigenvalues of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q + \mathrm{diag}\binom{1}{1} \otimes z$ are bounded by μ we conclude that the matrix $\mu\boldsymbol{E} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q - \mathrm{diag}\binom{1}{1} \otimes z$ is positive semidefinite, i.e.,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q \leq \mu\boldsymbol{E} - \mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix} \otimes z.$$

Therefore,

$$\mathcal{Q} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q \leq \frac{1}{2}\left(\mu\boldsymbol{E} - \mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix} \otimes z\right) = \mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix} \otimes y'.$$

Hence, y' is a feasible solution to $\mathrm{DSDP}'(Q)$. Furthermore, since $z \perp \mathbf{1}$ we obtain

$$\mathrm{DSDP}'(Q) \leq 2\langle \mathbf{1}, y' \rangle = \mu n = \mathrm{DSDP}''(Q),$$

as desired. $\square$

6. The Algorithmic Regularity Lemma and MAX-CUT

6.1. The Algorithmic Regularity Lemma

In this section we establish Theorem 8. The proof of Theorem 8 is conceptually similar to Szemerédi's original proof of the "dense" regularity lemma [105] and its adaptation for sparse graphs due to Kohayakawa [61] and Rödl (unpublished). A new aspect here is that we deal with a different (more general) notion of regularity; this requires various technical modifications of the previous arguments. More importantly, we present an *algorithm* for actually computing a regular partition of a sparse graph efficiently. Devising such an algorithm was actually posed as an open problem by Kohayakawa [61].

In order to find a regular partition efficiently, we crucially need an algorithm to check whether a pair of vertex sets is $(\epsilon, \boldsymbol{D})$-regular. In the next section, Section 6.1.1, we present the algorithm `Witness` that exhibits this feature and in Section 6.1.2, we will see how this can be used to construct the polynomial time algorithm `Regularise` that computes a partition satisfying **REG1** and **REG2** for a given graph G, provided that G satisfies the assumptions of Theorem 8. In particular, this shows that such a partition exist and thus prove Theorem 8.

In the following, let $0 < \epsilon < 10^{-7}$ be an arbitrarily small but fixed number, and $C \geq 1$ signifies an arbitrarily large but fixed number. In addition, we define a sequence $(t_k)_{k \geq 1}$ by

$$t_1 = \lceil 1/\epsilon^2 \rceil \quad \text{and} \quad t_{k+1} = \lceil 2200^2 C^2 t_k^6 2^{t_k} / \epsilon^{4(k+1)} \rceil. \tag{6.1}$$

Note that due to that choice we have

$$t_{k+1} \geq 2200 C t_k^{2.5}. \tag{6.2}$$

Further, let

$$k^* = \lceil 10^6 C^2 \epsilon^{-3} \rceil \quad \text{and} \quad \eta = \min \left\{ \frac{\epsilon^{8k^*}}{12800^2 t_{k^*}^6 C^4}, \frac{1}{t_{k^*}^2} \right\} \tag{6.3}$$

and choose $n_0 = n_0(C, \epsilon) > 0$ big enough. We let $G = (V, E)$ be a graph on $n = |V| > n_0$ vertices, and let $\boldsymbol{D} = (D_v)_{v \in V}$ be a sequence of rationals which satisfies $1 \leq D_v \leq n$ for all $v \in V$. We will always assume that G is $(C, \eta, \boldsymbol{D})$-bounded, and that $D(V) \geq \eta^{-1} n$.

6.1.1. The Procedure `Witness`

The subroutine `Witness` is given a graph G, a weight distribution $\boldsymbol{D}$, vertex sets A, B, and a number $\epsilon > 0$. `Witness` either outputs "yes", in which case (A, B) is $(\epsilon, \boldsymbol{D})$-regular in G, or "no". In the latter case the algorithm also produces a "witness of irregularity", i.e., a pair of sets $X^* \subset A$, $Y^* \subset B$ for which the regularity condition (1.3) is violated with ϵ replaced by $\epsilon/200$. `Witness` employs the algorithm `ApxCutNorm` from Theorem 69.

Algorithm 72. `Witness`$(G, \boldsymbol{D}, A, B, \epsilon)$

1. Set up the matrix $M = (m_{vw})_{(v,w)\in A\times B}$ with entries

$$m_{vw} = \begin{cases} 1 - \varrho(A,B)D_v D_w & \text{if } v, w \text{ are adjacent in } G, \\ -\varrho(A,B)D_v D_w & \text{otherwise.} \end{cases}$$

   Call `ApxCutNorm`(M) to compute sets $X \subset A$, $Y \subset B$ such that $|\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle| \geq \frac{3}{100}\|M\|_{\text{cut}}$.
2. If $|\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle| < \frac{3\epsilon}{100}\frac{D(A)D(B)}{D(V)}$, then return "yes".
3. If not, let $X' = A \setminus X$.
   - If $D(X) \geq \frac{3\epsilon}{100}D(A)$, then let $X^* = X$.
   - If $D(X) < \frac{3\epsilon}{100}D(A)$ and $|e(X', Y) - \varrho(A,B)D(X')D(Y)| > \frac{\epsilon D(A)D(B)}{100D(V)}$, set $X^* = X'$.
   - Otherwise, set $X^* = X \cup X'$.
4. Let $Y' = B \setminus Y$.
   - If $D(Y) \geq \frac{\epsilon}{200}D(B)$, then let $Y^* = Y$.
   - If $D(Y) < \frac{\epsilon}{200}D(B)$ and $|e(X^*, Y') - \varrho(A,B)D(X^*)D(Y')| > \frac{\epsilon D(A)D(B)}{200D(V)}$, set $Y^* = Y'$.
   - Otherwise, set $Y^* = Y \cup Y'$.
5. Answer "no" and output (X^*, Y^*) as an $(\epsilon/200, \boldsymbol{D})$-witness.

Lemma 73. *Suppose that $A, B \subset V$ are disjoint.*

1. *If `Witness`$(G, \boldsymbol{D}, A, B, \epsilon)$ answers "yes", then the pair (A, B) is $(\epsilon, \boldsymbol{D})$-regular.*
2. *If the answer is "no", then (A, B) is not $(\epsilon/200, \boldsymbol{D})$-regular. In this case `Witness` outputs an $(\epsilon/200, \boldsymbol{D})$-witness, i.e., a pair (X^*, Y^*) of subsets $X^* \subset A$, $Y^* \subset B$ such that*
$$D(X^*) \geq \frac{\epsilon}{200}D(A), D(Y^*) \geq \frac{\epsilon}{200}D(B)$$
   and
$$|e(X^*, Y^*) - \varrho(A,B)D(X^*)D(Y^*)| > \frac{\epsilon}{200} \cdot \frac{D(A)D(B)}{D(V)}.$$

Moreover, there exist a function f and a polynomial Π such that the running time of `Witness` is bounded by $f(C, \epsilon) \cdot \Pi(\langle \boldsymbol{D} \rangle)$.

Proof. Note that for any two subsets $S \subset A$ and $T \subset B$ we have

$$\langle M\mathbf{1}_S, \mathbf{1}_T\rangle = e(S,T) - \varrho(A,B)D(S)D(T).$$

Therefore, if the sets $X \subset A$ and $Y \subset B$ computed by `ApxCutNorm` are such that

$$|\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle| < \frac{3\epsilon}{100}\frac{D(A)D(B)}{D(V)}$$

then by Theorem 69 we have

$$|e(S,T) - \varrho(A,B)D(S)D(T)| \le \|M\|_{\text{cut}} \le \frac{100}{3}|\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle| < \epsilon\frac{D(A)D(B)}{D(V)}$$

for all $S \subset A$ and $T \subset B$. Thus, if `Witness` answers "yes" then the pair (A,B) is $(\epsilon, \boldsymbol{D})$-regular.

One the other hand, if the algorithm `ApxCutNorm` yields sets X, Y which satisfy $\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle \ge \frac{3\epsilon}{100}\frac{D(A)D(B)}{D(V)}$ then `Witness` has to guarantee that the output pair (X^*, Y^*) is an $(\epsilon/200, \boldsymbol{D})$-witness.

Indeed, if

$$D(X) \ge \frac{3\epsilon}{100}D(A) \quad \text{and} \quad D(Y) \ge \frac{\epsilon}{200}D(B)$$

then (X,Y) actually is an $(\epsilon/200, \boldsymbol{D})$-witness. However, as `ApxCutNorm` does not guarantee any lower bound on $D(X)$ and $D(Y)$ let assume first that

$$D(X) < \frac{3\epsilon}{100}D(A) \quad \text{and} D(Y) \ge \frac{\epsilon}{200}D(B).$$

Then Step 3 of `Witness` sets $X' = A \setminus X$ and we have $D(X') \ge \frac{3}{100}D(A)$. If X' itself satisfies

$$|e(X',Y) - \varrho(A,B)D(X')D(Y)| > \frac{\epsilon D(A)D(B)}{100D(V)}$$

then (X',Y) obviously is an $(\epsilon/200, \boldsymbol{D})$-witness. Otherwise, by the triangle inequality, we deduce

$$\left|e(X \cup X', Y) - e(A,B)\frac{D(X \cup X')D(Y)}{D(A)D(B)}\right| \ge \frac{2\epsilon}{100}\frac{D(A)D(B)}{D(V)}$$

and thus, $(X \cup X', Y)$ is an $(\epsilon/200, \boldsymbol{D})$-witness.

In the case $D(X) < \frac{3\epsilon}{100}D(A)$ and $D(Y) < \frac{\epsilon}{200}D(B)$ we simply repeat the argument for Y, and hence `Witness` outputs an $(\epsilon/200, \boldsymbol{D})$-witness for (A,B).

The running time of `Witness` is clearly dominated by Step 1, i.e., the execution of `ApxCutNorm`. By Theorem 69 the running time of `ApxCutNorm` is polynomial in the encoding length of the input matrix. Moreover, the construction of M in Step 1 shows that its encoding length is of the form $f(C,\epsilon) \cdot \Pi(\langle \boldsymbol{D}\rangle)$ for a certain function f and a polynomial Π, as claimed. $\square$

6.1.2. The Algorithm `Regularise`

To compute the regular partition of the input graph G the algorithm `Regularise` starts with an arbitrary initial partition $\mathcal{P}^1 = \{V_i^1 : i = 0, 1, \ldots, s_1\}$ such that each class V_i^1 $(i = 1, \ldots, s_1)$ has a "decent" weight $D(V_i^1)$. In the subsequent steps, `Regularise` computes a sequence $(\mathcal{P}^k)$ of partitions such that $\mathcal{P}^{k+1}$ is a "more regular" refinement of $\mathcal{P}^k$. The algorithm halts as soon as it can verify that $\mathcal{P}^k$ satisfies both **REG1** and **REG2** of Theorem 8. To this end `Regularise` applies the subroutine `Witness` to each pair (V_i^k, V_j^k) of the current partition $\mathcal{P}^k$. By Lemma 73 this yields a set $\mathcal{L}^k$ of pairs (i, j) such that all (V_i^k, V_j^k) with $(i, j) \notin \mathcal{L}^k$ are $(\epsilon, \boldsymbol{D})$-regular. Hence, $\mathcal{P}^k$ satisfies **REG2** as soon as $\sum_{(i,j)\in\mathcal{L}^k} D(V_i^k)D(V_j^k) < \epsilon D(V)^2$. In this case the algorithm `Regularise` stops and outputs $\mathcal{P}^k$. As we will see, all partitions $\mathcal{P}^k$ satisfy **REG1** by construction. Consequently, `Regularise` stops with a desired regular partition.

Algorithm 74. `Regularise`$(G, C, \boldsymbol{D}, \epsilon)$

1. Fix an arbitrary partition $\mathcal{P}^1 = \{V_i^1 : 0 \le i \le s_1\}$ for some $s_1 \le t_1$ with the property
   - $D(V)/t_1 - \max_{v\in V} D_v < D(V_i^1) \le D(V)/t_1$ for all $1 \le i \le s_1$ and
   - $D(V \setminus (\bigcup_{i\in[s_1]} V_i^1)) \le D(V)/t_1$.

   Set $V_0^1 = V \setminus \bigcup_{i\in[s_1]} V_i^1$ and set $k^* = \lceil 1000^2 C^2 \epsilon^{-3} \rceil$.
2. For $k = 1, 2, 3, \ldots, k^*$ do
3. Initially, let $\mathcal{L}^k = \emptyset$.
   For each pair (V_i^k, V_j^k) $(i < j)$ of classes of partition $\mathcal{P}^k$
4. call the procedure `Witness`$(G, \boldsymbol{D}, V_i^k, V_j^k, \epsilon)$.
   If it answers "no" and outputs an hence $(\epsilon/200, \boldsymbol{D})$-witness (X_{ij}^k, X_{ji}^k) for (V_i^k, V_j^k) then add (i, j) to $\mathcal{L}^k$.
5. If $\sum_{(i,j)\in\mathcal{L}^k} D(V_i^k)D(V_j^k) < \epsilon(D(V))^2$, then output the partition $\mathcal{P}^k$ and halt.
6. Else construct a refinement $\mathcal{P}^{k+1}$ of $\mathcal{P}^k$ as follows:
   - First construct the unique minimal partition $\mathcal{C}^k$ of $V \setminus V_0^k$, which refines $\{X_{ij}^k, V_i \setminus X_{ij}^k\}$ for every $i = 1, \ldots, s_k$ and every $j \ne i$. More precisely, we define the equivalence relation $\equiv_i^k$ on V_i by letting $u \equiv_i^k v$ iff for all j such that $(i, j) \in \mathcal{L}_k$ it is true that $u \in X_{ij}^k \Leftrightarrow v \in X_{ij}^k$ and we let $\mathcal{C}^k$ be the set of all equivalence classes of the relations $\equiv_i^k$ $(1 \le i \le s_k)$.
   - Set $\alpha_k = \epsilon^{4(k+1)}/(2200^2 C^2 t_k^6 2^{t_k})$ and split each vertex class of $\mathcal{C}^k$ into blocks with weights between $\alpha_k D(V)$ and $\alpha_k D(V) + \max_{v\in V} D_v$ and possibly one exceptional block of smaller weight. More precisely, construct a refinement $\mathcal{C}_*^k = \{V_{0,1}^{k+1}, \ldots, V_{0,r_k}^{k+1}, V_1^{k+1}, \ldots, V_{s_{k+1}}^{k+1}\}$ of $\mathcal{C}^k$ such that:
     - $r_k \le |\mathcal{C}^k| \le s_k 2^{s_k}$,
     - $D(V_{0,q}^{k+1}) < \alpha_k D(V)$ for all $q \in [r_k]$, and
     - $\alpha_k D(V) \le D(V_i^{k+1}) < \alpha_k D(V) + \max_{v\in V} D_v$ for all $i \in [s_{k+1}]$.
   - Let $V_0^{k+1} = V_0^k \cup \bigcup_{q\in[r_k]} V_{0,q}^{k+1}$ and set $\mathcal{P}^{k+1} = \{V_i^{k+1} : 0 \le i \le s_{k+1}\}$.

Step 6 is the central step of the algorithm. In the first part of that step we construct a joint refinement of the previous partition $\mathcal{P}^k$ and all the witnesses of irregularity (X_{ij}^k, X_{ji}^k) discovered in Step 4. Similarly as in the original proof of Szemerédi's it will turn out that a bounded parameter (the so-called index defined below) of the partition $\mathcal{C}^k$ increases by $\Omega(\epsilon^3)$ compared to $\mathcal{P}^k$. Since $\mathcal{P}_k$ consists of s_k classes and for every $i = 1, \ldots, s_k$ there are at most $s_k - 1$ witness sets X_{ij} $(j \neq i)$, the refinement $\mathcal{C}^k$ contains at most $s_k 2^{s_k - 1} < s_k 2^{s_k}$ vertex classes. In the second part of Step 6 we split the classes of $\mathcal{C}^k$ into pieces of almost equal weight. Here for each class of $\mathcal{C}^k$ we may get one class of left-over vertices $V_{0,q}^k$ of smaller weight, which together with V_0^k form the new exceptional class V_0^{k+1}. Due to the construction in Step 6, the bound $s_1 \leq t_1$, and (6.1) for any $k \geq 0$ the partition $\mathcal{P}^{k+1}$ consist of at most

$$s_{k+1} + 1 \leq \lceil 2200^2 C^2 t_k^6 2^{t_k} / \epsilon^{4(k+1)} \rceil = t_{k+1}$$

classes. Moreover, our choice (6.3) of η and the construction in Step 1 ensure that

$$\epsilon^2 D(V) \geq D(V_i^{k+1}) \geq \sqrt{\eta} D(V) \text{ for all } 1 \leq i \leq s_{k+1} \tag{6.4}$$

for every $k < k^*$ (since in Step 6 we put all vertex classes of "extremely small" weight into the exceptional class). Furthermore, due to $r_i \leq s_i 2^{s_i}$, $s_i \leq t_i$, and $\epsilon < 1/2$ we have

$$\begin{aligned} D(V_0^{k+1}) &\leq D(V_0^1) + \sum_{i=2}^{k+1} r_i \frac{\epsilon^{4(i+1)}}{2200^2 C^2 t_k^6 2^{t_k}} D(V) \\ &\leq \frac{D(V)}{t_1} + D(V) \sum_{i=2}^{k+1} \epsilon^{2i} \leq \frac{\epsilon^2}{1 - \epsilon^2} D(V) \leq \epsilon D(V) \,. \end{aligned}$$

In effect, $\mathcal{P}^{k+1}$ always satisfies **REG1**, as **REG1(c)** is ensured by Step 6.

Thus, to complete the proof of Theorem 8 it suffices to show that Step 5 of `Regularise` will output a partition $\mathcal{P}^k$ for some $k \leq k^*$. More precisely, we have to show that for every input graph G there exists a $k \leq k^*$ such that $\sum_{(i,j) \in \mathcal{L}^k} \boldsymbol{D}(V_i^k) \boldsymbol{D}(V_j^k) < \epsilon (\boldsymbol{D}(V))^2$. To show this, we use, as in the original proof of Szemerédi [105], the concept of the *index* of a partition $\mathcal{P} = \{V_i : 0 \leq i \leq s\}$ and define

$$\operatorname{ind}(\mathcal{P}) = \sum_{1 \leq i < j \leq s} \varrho(V_i, V_j)^2 D(V_i) D(V_j) = \sum_{1 \leq i < j \leq s} \frac{e(V_i, V_j)^2}{D(V_i) D(V_j)}.$$

Note that we do *not* take into account the (exceptional) class V_0 here. Using the boundedness-condition, we derive the following.

Proposition 75. *Given a $(C, \eta, \boldsymbol{D})$-bounded graph $G = (V, E)$ and given a partition $\mathcal{P} = \{V_i \colon 0 \leq 1 \leq t\}$ of V which satisfies $D(V_i) \geq \eta D(V)$ for all $i \in \{1, \ldots, t\}$, then $0 \leq \operatorname{ind}(\mathcal{P}) \leq C^2$.*

Proof. Since $D(V_i) \geq \eta D(V)$ for every $i \in \{1, \ldots, t\}$ we deduce from the boundedness of G that

$$\mathrm{ind}(\mathcal{P}) = \sum_{1\le i<j\le s} \frac{e(V_i,V_j)^2}{D(V_i)D(V_j)} \le \sum_{1\le i<j\le s} \frac{Ce(V_i,V_j)}{D(V)} \le C\frac{e(V,V)}{D(V)} \le C^2,$$

which proves the proposition. □

Proposition 75 and (6.4) imply that $\mathrm{ind}(\mathcal{P}^k) \le C^2$ for all k. In addition, since `Regularise` obtains $\mathcal{P}^{k+1}$ by refining $\mathcal{P}^k$ according to the witnesses of irregularity computed by `Witness`, the index of $\mathcal{P}^{k+1}$ is actually considerably larger than the index of $\mathcal{P}^k$. More precisely, the following is true.

Lemma 76. *If* $\sum_{(i,j)\in\mathcal{L}^k} D(V_i^k)D(V_j^k) \ge \epsilon(D(V))^2$, *then*

$$\mathrm{ind}(\mathcal{P}^{k+1}) \ge \mathrm{ind}(\mathcal{P}^k) + \epsilon^3/8.$$

The proof of Lemma 76 is deferred to the next section, Section 6.1.3.

We close this section by pointing out that Propositions 75 and Lemma 76 readily imply that `Regularise` will terminate and output a feasible partition $\mathcal{P}^k$ for some $k \le k^*$. Moreover, the dominant contribution to the running time of `Regularise` stems from the execution of the subroutine `Witness`, which gets called at most $O(k^* t_{k^*}^2)$ times. By Lemma 73 each execution takes time $f(C,\epsilon)\cdot\Pi(\langle \boldsymbol{D}\rangle)$ for a certain function f and a polynomial Π. Hence, the total running time of `Regularise` is bounded by $f^*(C,\epsilon)\cdot\Pi(\langle \boldsymbol{D}\rangle)$, where $f^*(C,\epsilon) = O(k^* t_{k^*}^2)\cdot f(C,\epsilon)$.

6.1.3. Proof of Lemma 76

As mentioned before, the proof of Lemma 76 follows the lines of the original proof of Szemerédi [105] with the main differences resulting from the somewhat different concept of regularity.

For the proof we will need the following technical proposition. Its proof is straightforward and we omit it here.

Proposition 77. *Let* $1/5 > \delta > 0$, $\eta > 0$, $C \ge 1$, *and* $\boldsymbol{D} = (D_v)_{v\in V}$ *be a sequence of rationals with* $1 \le D_v \le n$ *for all* $v \in V$. *Let* $G = (V,E)$ *be a* $(C,\eta,\boldsymbol{D})$*-bounded graph and* $A, B \subset V$ *be disjoint subsets of* V *with* $D(A), D(B) \ge \sqrt{\eta}D(V)$. *If* $A' \subset A$ *and* $B' \subset B$ *satisfy* $D(A\setminus A') < \delta D(A)$ *and* $D(B\setminus B') < \delta D(B)$, *then*

$$\left|\frac{e(A,B)}{D(A)D(B)} - \frac{e(A',B')}{D(A')D(B')}\right| \le \frac{(7\delta+4\sqrt{\eta})C}{D(V)}$$
$$\left|\frac{e^2(A,B)}{D(A)D(B)} - \frac{e^2(A',B')}{D(A')D(B')}\right| \le (21\delta+9\sqrt{\eta})C^2.$$

For two partitions $\mathcal{P}' = \{V_j' \colon 0 \le j \le s\}$ and $\mathcal{P} = \{V_i \colon 0 \le i \le t\}$ we say $\mathcal{P}'$ *almost refines* $\mathcal{P}$, if for every $j \in [s]$ there exists an $i \in [t]$ such that $V_j' \subset V_i$. Note that an almost refinement may not be a refinement, since V_0' could be a proper superset of V_0. However, it is easily seen that the index of an almost refinement cannot drop.

Proposition 78. *Let $\mathcal{P}' = \{V_j' \colon 0 \le j \le s\}$ and $\mathcal{P} = \{V_i \colon 0 \le i \le t\}$ be two partitions of V. If $\mathcal{P}'$ almost refines $\mathcal{P}$, then* $\mathrm{ind}(\mathcal{P}') \ge \mathrm{ind}(\mathcal{P})$.

Proof. For $V_i \in \mathcal{P}$, $i \in [t]$ let $I_i = \{j \colon V_j' \in \mathcal{P}', V_j' \subset V_i\}$. Then, using the Cauchy-Schwarz-inequality, we conclude

$$\begin{aligned}\mathrm{ind}(\mathcal{P}') &= \sum_{1 \le i < j \le s} \frac{e^2(V_i', V_j')}{D(V_i')D(V_j')} \ge \sum_{1 \le k < l \le t} \sum_{\substack{i \in I_k \\ j \in I_l}} \frac{e^2(V_i', V_j')}{D(V_i')D(V_j')} \\ &\ge \sum_{1 \le k < l \le t} \frac{\left(\sum_{i \in I_k, j \in I_l} e(V_i', V_j')\right)^2}{\sum_{i \in I_k, j \in I_l} D(V_i')D(V_j')} = \sum_{1 \le k < l \le t} \frac{e^2(V_k, V_l)}{D(V_k)D(V_l)} = \mathrm{ind}(\mathcal{P}),\end{aligned}$$

hence the proposition follows. □

With these auxiliary statements at hand we are now prepared for the proof of the main Lemma of this section.

Proof of Lemma 76. Recall that we assumed $\epsilon < 10^{-7}$. Let $K \subset V$ be the union of the equivalence classes with negligible weight; more precisely, in view of Step 6 we set

$$K = \bigcup_{q \in [r_k]} V_{0,q}^{k+1}.$$

Note that due to $r_k \le s_k 2^{s_k}$ and $s_k \le t_k$ we have

$$D(K) \le r_k \frac{\epsilon^{4(k+1)}}{2200^2 C^2 t_k^6 2^{t_k}} D(V) \le \frac{\epsilon^{4(k+1)}}{2200^2 C^2 t_k^5} D(V). \tag{6.5}$$

Now let $\mathcal{P}' = \{V_i' \colon 0 \le i \le s_k\}$ be the partition given by

$$V_i' = \begin{cases} V_0^k \cup K & \text{if } i = 0, \\ V_i^k \setminus K & \text{otherwise.} \end{cases}$$

To show the index increment $\mathrm{ind}(\mathcal{P}^{k+1}) \ge \mathrm{ind}(\mathcal{P}^k) + \epsilon^3/1000^2$ we will proceed in two steps. In the first step we will compare the index of $\mathcal{P}'$ to the index of $\mathcal{P}^k$.

Claim 79. $|\mathrm{ind}(\mathcal{P}^k) - \mathrm{ind}(\mathcal{P}')| \le \epsilon^4$.

The second step will reveal the index increment of $\mathcal{P}^{k+1}$ compared to $\mathcal{P}'$.

Claim 80. $\mathrm{ind}(\mathcal{P}^{k+1}) \ge \mathrm{ind}(\mathcal{P}') + \epsilon^3/800^2$.

As $\epsilon < 10^{-7}$, we obtain $\mathrm{ind}(\mathcal{P}^{k+1}) \ge \mathrm{ind}(\mathcal{P}^k) + \epsilon^3/1000^2$. □

Proof of Claim 79. Let (V_i^k, V_j^k) be a pair of partition classes of $\mathcal{P}^k$ and let $V_i' = V_i^k \setminus K$ and $V_j' = V_j^k \setminus K$. Note that due to $D(V_i^k) \geq \epsilon^{4k} D(V)/t_k^3$ and (6.5) we have

$$D(V_i') \geq D(V_i^k) - D(K) \geq (1 - \tfrac{\epsilon^4}{42C^2 t_k^2}) D(V_i^k).$$

Analogously $D(V_j') \geq (1 - \epsilon^4/(42C^2 t_k^2))\, D(V_j^k)$ holds. In effect, using Proposition 77 we get

$$\left| \frac{e^2(V_i', V_j')}{D(V_i')D(V_j')} - \frac{e^2(V_i^k, V_j^k)}{D(V_i^k)D(V_j^k)} \right| \leq \frac{\epsilon^4}{2t_k^2} + 9\sqrt{\eta}C^2 \overset{(6.3)}{\leq} \frac{\epsilon^4}{t_k^2}.$$

Consequently

$$|\mathrm{ind}(\mathcal{P}^k) - \mathrm{ind}(\mathcal{P}')| \leq \sum_{1 \leq i < j \leq s_k} \left| \frac{e^2(V_i^k, V_j^k)}{D(V_i^k)D(V_j^k)} - \frac{e^2(V_i', V_j')}{D(V_i')D(V_j')} \right| \leq \epsilon^4,$$

as claimed. □

Proof of Claim 80. Let (V_i^k, V_j^k) be an irregular pair and for notational convenience let $(A, B) = (V_i^k \setminus K, V_j^k \setminus K)$. Furthermore, let (X_{ij}^k, X_{ji}^k) be an $(\epsilon/200, \boldsymbol{D})$-witness. Then, for $X = X_{ij}^k \setminus K \subset A$ and $Y = X_{ji}^k \setminus K \subset B$, we have due to Proposition 77

$$\begin{aligned}
&\left| \frac{e(X,Y)}{D(X)D(Y)} - \frac{e(A,B)}{D(A)D(B)} \right| \\
&\qquad \geq \frac{\epsilon}{200} \frac{D(A)D(B)}{D(X_{ij}^k)D(X_{ji}^k)D(V)} - \frac{\frac{7\epsilon^2}{2200^2} + \frac{7 \cdot 200\epsilon}{2200^2} + 8\sqrt{\eta}C}{D(V)} \\
&\qquad \geq \frac{\epsilon}{400} \frac{D(A)D(B)}{D(X)D(Y)D(V)} - \frac{\epsilon}{1600 D(V)} - \frac{\epsilon}{1600 D(V)} \\
&\qquad \geq \frac{\epsilon}{800} \frac{D(A)D(B)}{D(X)D(Y)D(V)}\,.
\end{aligned} \tag{6.6}$$

Thus, (X, Y) 'witnesses' that (A, B) is not $(\epsilon/800, \boldsymbol{D})$-regular.

Now we will use Lemma 63 to prove $\mathrm{ind}(\mathcal{P}^{k+1}) \geq \mathrm{ind}(\mathcal{P}') + \epsilon^3/4$. To this end, let $I = A \times B$ and for all $(u, v) \in I$ let

$$\sigma_{uv} = \frac{D_u D_v}{D(A)D(B)} \quad \text{and} \quad \varrho_{uv} = \varrho(V^{k+1}(u), V^{k+1}(v))$$

where $V^{k+1}(x)$ denote the partition class $V_i^{k+1} \in \mathcal{P}^{k+1}$ such that $x \in V_i^{k+1}$. Then $\sum_{(u,v) \in I} \sigma_{uv} = 1$ and

$$\sum_{(u,v) \in I} \sigma_{uv} \varrho_{uv} = \sum_{(u,v) \in I} \frac{D_u D_v}{D(A)D(B)} \frac{e(V^{k+1}(u), V^{k+1}(v))}{D(V^{k+1}(u))D(V^{k+1}(v))} = \varrho(A, B).$$

Moreover, let $J = X \times Y$ and $\sigma_J = \sum_{(u,v)\in J} \sigma_{uv} = \frac{D(X)D(Y)}{D(A)D(B)}$. Then we have

$$\begin{aligned}\frac{1}{\sigma_J}\sum_{(u,v)\in J}\sigma_{uv}\varrho_{uv} &= \frac{D(A)D(B)}{D(X)D(Y)}\sum_{\substack{V_i^{k+1}\subset X\\ V_j^{k+1}\subset Y}}\sum_{\substack{u\in V_i^{k+1}\\ v\in V_j^{k+1}}}\frac{D_uD_v}{D(A)D(B)}\varrho(V_i^{k+1},V_j^{k+1})\\ &= \frac{e(X,Y)}{D(X)D(Y)} = \varrho(X,Y) = \varrho(A,B)+\nu\end{aligned}$$

for some $|\nu| \geq \epsilon D(A)D(B)/(800D(X)D(Y)D(V))$ due to (6.6).

Hence, from the Cauchy-Schwarz-inequality (Lemma 63) we deduce

$$\begin{aligned}&\frac{1}{D(A)D(B)}\sum_{\substack{V_i^{k+1}\subset A\\ V_j^{k+1}\subset B}}\varrho^2(V_i^{k+1},V_j^{k+1})D(V_i^{k+1})D(V_j^{k+1})\\ &\qquad= \sum_{u,v\in I}\frac{D_uD_v}{D(A)D(B)}\varrho^2(V^{k+1}(u),V^{k+1}(v)) = \sum_{(u,v)\in I}\sigma_{uv}\varrho_{uv}^2\\ &\qquad\geq \varrho^2(A,B)+\left(\frac{\epsilon D(A)D(B)}{800D(X)D(Y)D(V)}\right)^2\frac{D(X)D(Y)}{D(A)D(B)}\\ &\qquad\geq \frac{1}{D(A)D(B)}\left(\varrho^2(A,B)D(A)D(B)+\frac{\epsilon^2D(A)D(B)}{800^2D^2(V)}\right).\end{aligned}$$

From the last inequality we infer the amount of the index increment on the irregular pair (A, B). So, in view of Proposition 78, after summing over all pairs we get

$$\operatorname{ind}(\mathcal{P}^{k+1}) - \operatorname{ind}(\mathcal{P}') \geq \sum_{(i,j)\in\mathcal{L}^k}\frac{\epsilon^2}{800^2}\frac{D(A)D(B)}{D^2(V)} \geq \frac{\epsilon^3}{800^2},$$

as stated in the claim. □

6.2. An Application: MAX-CUT

As an application of Theorem 8 and, in particular, the polynomial time algorithm `Regularise` for computing a regular partition, we obtain the following algorithm for approximating the maximum cut of a graph $G = (V, E)$ that satisfies the assumptions of Theorem 9.

Algorithm 81. `ApxMaxCut`$(G, C, \boldsymbol{D}, \delta)$
Input: A $(C, \eta, \boldsymbol{D})$-bounded graph $G = (V, E)$ and $\delta > 0$.
Output: A cut $(S, \bar{S})$ of G.

1. Use the algorithm `Regularise` to compute $\epsilon = \frac{\delta}{400C}$-regular partition $\mathcal{P} = \{V_i : 0 \leq i \leq t\}$ of G.

2. Determine an optimal solution $(c_1^*, \ldots, c_t^*)$ to the optimisation problem

$$\max \sum_{i \neq j} \epsilon c_i(1 - \epsilon c_j)e(V_i, V_j) \text{ s.t. } \forall 1 \leq j \leq t : 0 \leq c_j \leq \epsilon^{-1},\ c_j \in \mathbb{N}.$$

3. For each $1 \leq i \leq t$ let $S_i \subset V_i$ such that $|D(S_i) - c_i^* \epsilon D(V_i)| \leq 2\epsilon D(V_i)$. Output $S = \bigcup_{i=1}^t S_i$ and $\bar{S} = V \setminus S$.

The basic insight behind `ApxMaxCut` is the following. If (V_i, V_j) is an $(\epsilon, \boldsymbol{D})$-regular pair of $\mathcal{P}$, then for any subsets $X, X' \subset V_i$ and $Y, Y' \subset V_j$ such that $D(X) = D(X')$ and $D(Y) = D(Y')$ the condition **REG2** ensures that

$$|e(X, Y) - e(X', Y')| \leq \frac{2\epsilon D(V_i)D(V_j)}{D(V)},$$

that is, the difference between $e(X, Y)$ and $e(X', Y')$ is negligible. In other words, as far as the number of edges is concerned, subsets that have the same weight are "interchangeable".

Therefore, to compute a good cut $(S, \bar{S})$ of G we just have to optimise the *proportion of weight* of each V_i that is to be put into S or into $\bar{S}$, but it does not matter which subset of V_i of this weight we choose. However, determining the optimal fraction of weight is still a somewhat involved (essential continuous) optimisation problem. Hence, in order to discretise this problem, we chop each V_i into at most ϵ^{-1} chunks of weight $\epsilon D(V_i)$. Then, we just have to determine the number c_i of chunks of each V_i that we join to S. This is exactly the optimisation problem detailed in Step 2 of `ApxMaxCut`.

Observe that the time required to solve this problem is *independent* of n. Indeed, the number t of classes of $\mathcal{P}$ is bounded by a number independent of n, and the number $\lceil \epsilon^{-1} \rceil + 1$ of choices for each c_i does not depend on n either, hence, Step 2 has a *constant* running time.

In addition, Step 3 can be implemented to perform in linear time, because $S_i \subset V_i$ can be *any* subset that satisfies the condition stated in Step 3. Thus, the total running time of `ApxMaxCut` is polynomial.

To prove that `ApxMaxCut` does indeed guarantee an approximation within an additive $\delta D(V)$, we compare the maximum cut of G with the optimal solution μ^* of the optimisation problem from Step 2, i.e.,

$$\mu^* = \max \sum_{i,j} \epsilon c_i(1 - \epsilon c_j)e(V_i, V_j) \qquad (6.7)$$
$$\text{s.t. } \forall 1 \leq j \leq t : 0 \leq c_j \leq \epsilon^{-1},\ c_j \in \mathbb{N}.$$

To this end, we say that a cut $(T, \bar{T})$ of G is *compatible* with a feasible solution $(c_1, \ldots, c_t)$ to the optimisation problem (6.7) if

$$|D(T \cap V_i) - c_i \epsilon D(V_i)| \leq 2\epsilon D(V_i).$$

Lemma 82. *Suppose that* $(T, \bar{T})$ *is compatible with the feasible solution* $(c_1, \dots, c_t)$ *of (6.7). Moreover, let*

$$\mu = \sum_{i,j} \epsilon c_i (1 - \epsilon c_j) e(V_i, V_j)$$

be the objective function value corresponding to $(c_1, \dots, c_t)$. *Then we have*

$$|e(T, \bar{T}) - \mu| \leq \frac{\delta}{8} D(V).$$

Proof. Set $T_i = T \cap V_i$ and $\bar{T}_i = V_i \setminus T_i$, so that

$$e(T, \bar{T}) = \sum_{i \neq j} e(T_i, \bar{T}_j) + \sum_{i=0}^{t} e(T_i, \bar{T}_i)$$

and let $\mu_{ij} = \epsilon c_i (1 - \epsilon c_j) e(V_i, V_j)$ $(1 \leq i, j \leq t)$. Moreover, let $\mathcal{L}$ be the set of all pairs (i, j) such that (V_i, V_j) is not $(\epsilon, \boldsymbol{D})$-regular. Then **REG 2** and the $(C, \eta, \boldsymbol{D})$-boundedness of G imply that

$$\begin{aligned}
\sum_{(i,j) \in \mathcal{L}} \mu_{ij} \leq \sum_{(i,j) \in \mathcal{L}} e(V_i, V_j) &\leq \sum_{(i,j) \in \mathcal{L}} \frac{C D(V_i) D(V_j)}{D(V)} \\
&\leq C \epsilon D(V) = \frac{\delta}{400} D(V), \qquad (6.8) \\
\sum_{(i,j) \in \mathcal{L}} e(T_i, \bar{T}_j) \leq \sum_{(i,j) \in \mathcal{L}} e(V_i, V_j) &\leq \frac{\delta}{400} D(V).
\end{aligned}$$

Furthermore, since $D(V_0) \leq \epsilon D(V)$ and $C \geq 1$ we have

$$e(T_0, \bar{T}) + e(\bar{T}_0, T) \leq D(V_0) \leq \epsilon D(V) \leq \frac{\delta}{400} D(V),$$

and as $D(V_i) \leq \epsilon D(V)$ for all i, the $(C, \eta, \boldsymbol{D})$-boundedness condition yields

$$\sum_{i=1}^{t} e(T_i, \bar{T}_i) \leq \sum_{i=1}^{t} \frac{C D(V_i)^2}{D(V)} \leq C \epsilon D(V) = \frac{\delta}{400} D(V).$$

In addition, let

$$\mathcal{S} = \{(i, j) : i, j > 0,\ i \neq j \wedge (i, j) \notin \mathcal{L} \wedge (D(T_i) < \epsilon D(V_i) \vee D(\bar{T}_j) < \epsilon D(V_j))\}.$$

We shall prove below that

$$\left| \mu_{ij} - e(T_i, \bar{T}_j) \right| < 5 \epsilon e(V_i, V_j) + \epsilon \frac{D(V_i) D(V_j)}{D(V)} \qquad (6.9)$$

for all $(i,j) \notin (\mathcal{L} \cup \mathcal{S})$, $i,j > 0$, $i \neq j$, and

$$\sum_{(i,j)\in\mathcal{S}} \mu_{ij} + e(T_i, \bar{T}_j) < 6\epsilon D(V). \tag{6.10}$$

Combining (6.8)–(6.10), we thus obtain

$$\begin{aligned}
\left|e(T,\bar{T}) - \mu\right| &\leq \sum_{\substack{(i,j)\notin(\mathcal{L}\cup\mathcal{S})\\ i,j>0,\, i\neq j}} \left|\mu_{ij} - e(T_i,\bar{T}_j)\right| + \sum_{(i,j)\in(\mathcal{L}\cup\mathcal{S})} (\mu_{ij} + e(T_i,T_j)) \\
&\quad + e(T_0,\bar{T}) + e(\bar{T}_0,T) + \sum_{i=1}^{t} e(T_i,\bar{T}_i) \\
&\leq 6\epsilon D(V) + \frac{\delta}{200} D(V) + 6\epsilon D(V) + \frac{\delta}{400} D(V) + \frac{\delta}{400} D(V) \leq \frac{\delta}{8} D(V),
\end{aligned}$$

as desired.

To establish (6.9), consider a pair $(i,j) \notin (\mathcal{L} \cup \mathcal{S})$, $i \neq j$. Note that $D(T_i) \geq \epsilon D(V_i)$ and $D(\bar{T}_j) \geq \epsilon D(V_j)$ and (V_i, V_j) is $(\epsilon, \boldsymbol{D})$-regular, thus

$$\left| e(T_i, \bar{T}_j) - \frac{D(T_i)D(\bar{T}_j)}{D(V_i)D(V_j)} e(V_i, V_j) \right| < \frac{\epsilon D(V_i) D(V_j)}{D(V)}. \tag{6.11}$$

Moreover, as $(T, \bar{T})$ is compatible with $(c_1, \ldots, c_t)$,

$$\left| \frac{D(T_i)}{D(V_i)} - \epsilon c_i \right| < 2\epsilon, \qquad \left| \frac{D(\bar{T}_j)}{D(V_j)} - (1 - \epsilon c_j) \right| < 2\epsilon, \tag{6.12}$$

and combining (6.11) and (6.12) yields (6.9).

Finally, to prove (6.10), consider an index i such that $D(T_i) < \epsilon D(V_i)$. Then we have $\sum_{j=1}^{t} e(T_i, \bar{T}_j) \leq D(T_i) < \epsilon D(V_i)$ and similarly, if $D(\bar{T}_j) < \epsilon D(V_j)$ we obtain $\sum_{i=1}^{t} e(T_i, \bar{T}_j) < \epsilon D(V_j)$. Therefore,

$$\sum_{(i,j)\in\mathcal{S}} e(T_i, \bar{T}_j) < 2\epsilon D(V). \tag{6.13}$$

Further, if $D(T_i) < \epsilon D(V_i)$, then $c_i \leq 2$, because $(T, \bar{T})$ is compatible with $(c_1, \ldots, c_t)$. Thus $\sum_{j=1}^{t} \mu_{ij} \leq 2\epsilon \sum_j e(V_i, V_j) \leq 2\epsilon D(V_i)$. Analogously, if $D(\bar{T}_j) < \epsilon D(V_j)$, then $\sum_{i=1}^{t} \mu_{ij} \leq 2\epsilon D(V_j)$. Consequently,

$$\sum_{(i,j)\in\mathcal{S}} \mu_{ij} < 4\epsilon D(V). \tag{6.14}$$

Hence, (6.10) follows from (6.13) and (6.14). □

Proof of Theorem 9. Step 3 of `ApxMaxCut` ensures that $(S, \bar{S})$ is compatible with the vector $(c_1^*, \dots, c_t^*)$. Therefore, Lemma 82 yields

$$e(S, \bar{S}) \geq \mu^* - \frac{\delta}{8} D(V). \tag{6.15}$$

Further, let $(T, \bar{T})$ be a maximum cut of G. Then we can construct a feasible solution to (6.7) that is compatible with $(T, \bar{T})$ by letting

$$c_i = \left\lfloor \frac{D(T \cap V_i)}{\epsilon D(V_i)} \right\rfloor \qquad (1 \leq i \leq t).$$

Let $\mu = \sum_{i,j} \epsilon c_i (1 - \epsilon c_j) e(V_i, V_j)$ be the corresponding objective function value. Then Lemma 82 implies that

$$e(T, \bar{T}) \leq \mu + \frac{\delta}{8} D(V). \tag{6.16}$$

As μ^* is the optimal value of (6.7), we have $\mu^* \geq \mu$, and thus (6.15) and (6.16) yield $e(S, \bar{S}) \geq e(T, \bar{T}) - \frac{\delta}{4} D(V)$. Consequently, `ApxMaxCut` provides the desired approximation guarantee. □

7. Quasi-Random Graphs with General Degree Distributions

Before proving Theorem 10 we sketch the proof showing that for the case of sparse graphs and graphs with general degree distribution low discrepancy does not imply eigenvalue separation. To this end let $\epsilon > 0$ be given. Consider an n-vertex graph $G = (X\dot{\cup}Y, E)$ consisting of two components $G[X]$ and $G[Y]$ where $G[X]$ is a complete graph on $n^{1/2}$ vertices and $G[Y]$ is the random graph $G(m,p)$ with $m = n - n^{1/2}$ and $p = n^{-1/2}$. Since $e(G[X]) = o(e(G))$ is is easily seen that almost surely G has discrepancy $\epsilon > 0$ for sufficiently large n, i.e. for all $S \subset V$ we have $\left|e(S) - \frac{\mathrm{vol}(S)^2}{2\mathrm{vol}(V)}\right| < \epsilon \mathrm{vol}(V)$. However, since G consists of two components the spectrum of the adjacency matrix $A(G)$ and of the Laplacian $L(G)$ is simply the union of the spectra of $G[X]$ and $G[Y]$. Furthermore, $\lambda_{\max}(G[X]) = n^{1/2} - 1$ since $G[X]$ is an $n^{1/2}$-regular graph and it is well-known that the largest eigenvalue of $A(G(n,p))$ is $\Theta(pn)$ (cf.[36]), hence $\lambda_{\max}(A(G[Y])) = \Theta(n^{1/2})$. In particular, the adjacency matrix does not exhibits eigenvalue separation. Concerning the Laplacian, we note that the eigenvalue 0 appears twice in the spectrum of $L(G)$, thus, $L(G)$ also does not exhibits eigenvalue separation either. This answers a question of Chung and Graham [17] (see [76] for a connected counter example).

Furthermore, Theorem 10 states that there is a constant $\gamma > 0$ such that Disc$(\gamma\epsilon^2)$ implies ess-Eig(ϵ). This statement is best possible, up to the precise value of γ. This is seen from the following slightly more complicated probabilistic construction of a graph $G = (V,E)$ on n vertices that has Disc(10ϵ) but does not have ess-Eig$(0.01\sqrt{\epsilon})$. Assume that $\epsilon > 0$ is a sufficiently small number, and choose $n = n(\epsilon)$ sufficiently large. Moreover, let $X = \{1, \ldots, \sqrt{\epsilon}n\}$ and $\bar{X} = \{\sqrt{\epsilon}n + 1, \ldots, n\}$. Further, let $d = n/2$ and set

$$p_X = 1, \quad p_{X\bar{X}} = p_{\bar{X}X} = \frac{1 - 2\sqrt{\epsilon}}{2 - 2\sqrt{\epsilon}}, \quad p_{\bar{X}} = \frac{1 - 2\sqrt{\epsilon} + 2\epsilon}{2(1 - \sqrt{\epsilon})^2}.$$

Finally, let G be the random graph with vertex set $V = \{1, \ldots, n\}$ obtained as follows: any two vertices in X are adjacent, any two vertices in $\bar{X}$ are connected with probability $p_{\bar{X}}$ independently, and each possible X-$\bar{X}$ edge is present with probability $p_{X\bar{X}}$ independently. Thus, the vertices X form a clique. Moreover, the expected degree of each vertex is d. It easily seen that G satisfies Disc(10ϵ) almost surely. On the other hand, to see that G does not satisfy ess-Eig$(\sqrt{\epsilon}/2)$, let $\mathcal{E}$ be the matrix with entries

$$\mathcal{E}_{vw} = \begin{cases} 1 & \text{if } v,w \in X, \\ p_{X\bar{X}} & \text{if } (v,w) \in X \times \bar{X} \cup \bar{X} \times X, \\ p_{\bar{X}} & \text{if } v,w \in \bar{X}. \end{cases}$$

This matrix just comprises the probabilities that the vertices v, w are adjacent and using standard results on the eigenvalues of random matrices [36] we conclude that $\|\boldsymbol{E} - L(G) - d^{-1}\mathcal{E}\| = o(1)$. Let $W \subset \{1, \ldots, n\}$ with $|W| \geq (1 - 0.01\epsilon)n$ be an arbitrary set. Then $\|\boldsymbol{E} - L(G)_W - d^{-1}\mathcal{E}_W\| \leq \|\boldsymbol{E} - L(G) - d^{-1}\mathcal{E}\| = o(1)$. Therefore, in order to show that $\lambda_2(L(G)_W) < 1 - 0.01\sqrt{\epsilon}$ it suffices to prove that the matrix $\boldsymbol{E} - d^{-1}\mathcal{E}_W$ satisfies

$$\lambda_2(\boldsymbol{E} - d^{-1}\mathcal{E}_W) \leq 1 - \sqrt{\epsilon}/2. \tag{7.1}$$

Let $x = |X \cap W|$ and $\bar{x} = |\bar{X} \cap W|$. The matrix $d^{-1}\mathcal{E}_W$ has rank two, and the eigenvectors with non-zero eigenvalues lie in the space spanned by the vectors $\mathbf{1}_{X\cap W}$ and $\mathbf{1}_{\bar{X}\cap W}$. This implies that its non-zero eigenvalues coincide with those of the 2×2 matrix

$$\mathcal{E}_* = d^{-1} \cdot \begin{pmatrix} x & \bar{x} \cdot p_{X\bar{X}} \\ x \cdot p_{X\bar{X}} & \bar{x} \cdot p_{\bar{X}} \end{pmatrix}.$$

The smaller eigenvalue is at least $\sqrt{\epsilon}/(1 - \sqrt{\epsilon}) - \epsilon \geq \sqrt{\epsilon}/2$ from which we deduce $\lambda_2(\boldsymbol{E} - d^{-1}\mathcal{E}_W) \leq 1 - \sqrt{\epsilon}/2$.

7.1. From essential eigenvalue separation to low discrepancy

We prove the first part of Theorem 10. Suppose that $G = (V, E)$ is a graph that admits a set $W \subset V$ of volume $\mathrm{vol}(W) \geq (1 - \epsilon)\mathrm{vol}(V)$ such that the eigenvalues of the minor L_W of the normalised Laplacian satisfy

$$1 - \epsilon \leq \lambda_2 [L_W] \leq \lambda_{\max} [L_W] \leq 1 + \epsilon. \tag{7.2}$$

We may assume without loss of generality that $\epsilon < 0.01$. Our goal is to show that G has $\mathrm{Disc}(20\sqrt{\epsilon})$.

Let $\Delta = (\sqrt{d_v})_{v\in W} \in \mathbf{R}^W$ and let $\mathcal{L}_W$ denote the matrix whose vw'th entry is $(d_v d_w)^{-\frac{1}{2}}$ if v, w are adjacent, and 0 otherwise ($v, w \in W$), so that $L_W = \boldsymbol{E} - \mathcal{L}_W$. Further, let $\mathcal{M}_W = \mathrm{vol}(V)^{-1}\Delta\Delta^T - \mathcal{L}_W$. Then for all unit vectors $\xi \perp \Delta$ we have

$$L_W\xi - \xi = -\mathcal{L}_W\xi = \mathcal{M}_W\xi. \tag{7.3}$$

Moreover, for all $S \subset W$

$$|\langle \mathcal{M}_W \Delta_S, \Delta_S\rangle| = \left| \frac{\mathrm{vol}(S)^2}{\mathrm{vol}(V)} - 2e(S) \right|. \tag{7.4}$$

The key step of the proof is to derive the following bound.

Lemma 83. *We have $\|\mathcal{M}_W\| \leq 10\sqrt{\epsilon}$.*

If it were the case that $W = V$, then Lemma 83 would be immediate. For if $W = V$, then Δ is an eigenvector of $L = L_W$ with eigenvalue 0. Hence, the definition $\mathcal{M}_W = \|\Delta\|^{-2}\Delta\Delta^T - \boldsymbol{E} + L_W$ ensures that $\mathcal{M}_W\Delta = 0$. Moreover, for all $\xi \perp \Delta$

we have $\mathcal{M}_W\xi = (L_W - \boldsymbol{E})\xi$, whence (7.2) implies

$$\|\mathcal{M}_W\| \leq \max\{|\lambda_2\left[L_W\right] - 1|, |\lambda_{\max}\left[L_W\right] - 1|\} \leq \epsilon.$$

But of course generally W is a proper subset of V. In this case Δ is not necessarily an eigenvector of L_W. In fact, the smallest eigenvalue of L_W may be strictly positive. In order to prove Lemma 83 we will investigate the eigenvector ζ of L_W with the smallest eigenvalue $\lambda_1\left[L_W\right]$ and show that it is "close" to Δ. Then, we will use (7.2) to derive the desired bound on $\|\mathcal{M}_W\|$.

Proof of Lemma 83. Let ζ be a unit length eigenvector of L_W with eigenvalue $\lambda_1\left[L_W\right]$. There is a decomposition $\Delta = \|\Delta\| \cdot (s\zeta + t\chi)$, where $s^2 + t^2 = 1$ and $\chi \perp \zeta$ is a unit vector. Since

$$\langle L_W\Delta, \Delta\rangle = e(W, V \setminus W) \leq \mathrm{vol}(V \setminus W) \leq \epsilon\mathrm{vol}(V)$$

and $\|\Delta\|^2 = \mathrm{vol}(W) \geq (1-\epsilon)\mathrm{vol}(V) \geq 0.99\mathrm{vol}(V)$, we have

$$2\epsilon \geq \|\Delta\|^{-2} \langle L_W\Delta, \Delta\rangle = s^2 \langle L_W\zeta, \zeta\rangle + t^2 \langle L_W\chi, \chi\rangle. \tag{7.5}$$

As χ is perpendicular to the eigenvector ζ with eigenvalue $\lambda_1\left[L_W\right]$, Courant-Fischer (5.5) and (7.2) yield $\langle L_W\chi, \chi\rangle \geq \lambda_2\left[L_W\right] \geq \frac{1}{2}$. Hence, (7.5) implies $2\epsilon \geq t^2/2$. Consequently,

$$t^2 \leq 4\epsilon, \quad \text{and thus} \quad s^2 \geq 1 - 4\epsilon. \tag{7.6}$$

Let $\xi \perp \Delta$ be an unit vector and consider $\xi = x\zeta + y\eta$ where $\eta \perp \zeta$ is an unit vector. Because of $\zeta = s^{-1}\left(\frac{\Delta}{\|\Delta\|} - t\chi\right)$ we have $x = \langle\zeta, \xi\rangle = s^{-1}\left\langle\frac{\Delta}{\|\Delta\|}, \xi\right\rangle - \frac{t}{s}\langle\chi, \xi\rangle = -\frac{t}{s}\langle\chi, \xi\rangle$. Hence, (7.6) implies $x^2 \leq 5\epsilon$ and $y^2 \geq 1 - 5\epsilon$. Combining these two estimates with (7.2) and (7.3), we conclude that $\|\mathcal{M}_W\xi\| = \|L_W\xi - \xi\| \leq x(1 - \lambda_1\left[L_W\right]) + y\|L_W\eta - \eta\| \leq 3\sqrt{\epsilon}$. Hence, we have established that

$$\sup_{0 \neq \xi \perp \Delta} \frac{\|\mathcal{M}_W\xi\|}{\|\xi\|} \leq 3\sqrt{\epsilon}. \tag{7.7}$$

Furthermore, since $\|\Delta\|^2 = \mathrm{vol}(W)$, (7.4) implies

$$\begin{aligned}
\frac{|\langle\mathcal{M}_W\Delta, \Delta\rangle|}{\|\Delta\|^2} &= \left|\frac{\mathrm{vol}(W)}{\mathrm{vol}(V)} - \frac{2e(W)}{\mathrm{vol}(W)}\right| \\
&\leq \left|\frac{\mathrm{vol}(W)}{\mathrm{vol}(V)} - \frac{2e(W)}{\mathrm{vol}(V)}\right| + \left|\frac{2e(W)}{\mathrm{vol}(W)} - \frac{2e(W)}{\mathrm{vol}(V)}\right| \\
&= \frac{e(W, V \setminus W)}{\mathrm{vol}(V)} + \frac{2e(W)(\mathrm{vol}(V) - \mathrm{vol}(W))}{\mathrm{vol}(V)\mathrm{vol}(W)} \\
&\leq \frac{e(W, V \setminus W)}{\mathrm{vol}(V)} + \frac{\mathrm{vol}(V \setminus W)}{\mathrm{vol}(V)} \leq \frac{2\mathrm{vol}(V \setminus W)}{\mathrm{vol}(V)}.
\end{aligned} \tag{7.8}$$

Due to $\mathrm{vol}(W) \geq (1-\epsilon)\mathrm{vol}(V)$ we obtain $\|\Delta\|^{-2}\left|\langle\mathcal{M}_W\Delta, \Delta\rangle\right| \leq 2\epsilon$ and combined with (7.7), we conclude that $\|\mathcal{M}_W\| \leq 10\sqrt{\epsilon}$. □

Lemma 83 easily implies that G has Disc($20\sqrt{\epsilon}$). For let $R \subset V$ be arbitrary, set $S = R \cap W$, and let $T = R \setminus W$. Since $\|\Delta_S\|^2 = \mathrm{vol}(S) \leq \mathrm{vol}(V)$, Lemma 83 and (7.4) imply that

$$\left|\frac{\mathrm{vol}(S)^2}{2\mathrm{vol}(V)} - e(S)\right| \leq \|\mathcal{M}_W\| \cdot \|\Delta_S\|^2 \leq 10\sqrt{\epsilon}\mathrm{vol}(V). \tag{7.9}$$

Furthermore, as $\mathrm{vol}(W) \geq (1-\epsilon)\mathrm{vol}(V)$,

$$e(R) - e(S) \leq e(T) + e(S,T) \leq 2\mathrm{vol}(T) \leq \mathrm{vol}(V \setminus W) \leq \epsilon\mathrm{vol}(V),$$

and

$$\begin{aligned}\frac{\mathrm{vol}(R)^2 - \mathrm{vol}(S)^2}{2\mathrm{vol}(V)} &\leq \frac{\mathrm{vol}(T)^2}{2\mathrm{vol}(V)} + \frac{\mathrm{vol}(S)\mathrm{vol}(T)}{\mathrm{vol}(V)} \\ &\leq \frac{\mathrm{vol}(V \setminus W)^2}{2\mathrm{vol}(V)} + \mathrm{vol}(V \setminus W) \leq 2\epsilon\mathrm{vol}(V).\end{aligned}$$

These two estimates with (7.9) entails $\left|\frac{\mathrm{vol}(R)^2}{2\mathrm{vol}(V)} - e(R)\right| < 20\sqrt{\epsilon}\mathrm{vol}(V)$, i.e., G satisfies Disc($20\sqrt{\epsilon}$).

7.2. From low discrepancy to essential eigenvalue separation

In this section we establish the second part of Theorem 10. Let θ denote the constant from Theorem 68 and set $\gamma = 10^{-6}/\theta$. Assume that $G = (V,E)$ is a graph that has Disc($\gamma\epsilon^2$) for some $\epsilon < 0.001$. In addition, we may assume without loss of generality that G has no isolated vertices. Let d_v denote the degree of $v \in V$, let $n = |V|$, and set $\bar{d} = \mathrm{vol}(V)/n = \sum_{v\in V} d_v/n$. Our goal is to show that G has ess-Eig(ϵ). To this end, we introduce an additional property.

Cut(δ): We say G has Cut(δ) if the matrix $M = (m_{vw})_{v,w\in V}$ with entries

$$m_{vw} = \frac{d_v d_w}{\mathrm{vol}(V)} - e(v,w)$$

has cut norm $\|M\|_{\mathrm{cut}} < \delta \cdot \mathrm{vol}(V)$; here $e(v,w) = 1$ if $\{v,w\} \in E$ and $e(v,w) = 0$ otherwise.

Proposition 84. *For any $\delta > 0$ the following is true: if G satisfies* Disc(0.01δ)*, then G satisfies* Cut(δ).

Proof. Suppose that $G = (V,E)$ has Disc(0.01δ). We shall prove below that for any two

$S, T \subset V$

$$|\langle M\mathbf{1}_S, \mathbf{1}_T\rangle| \le 0.06\delta \mathrm{vol}(V) \text{ if } S \cap T = \emptyset, \tag{7.10}$$
$$|\langle M\mathbf{1}_S, \mathbf{1}_T\rangle| \le 0.02\delta \mathrm{vol}(V) \text{ if } S = T. \tag{7.11}$$

To see that (7.10) and (7.11) imply the assertion, consider arbitrary subsets $X, Y \subset V$. Letting $Z = X \cap Y$ and combining (7.10) and (7.11), we obtain

$$\begin{aligned}|\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle| &\le \left|\left\langle M\mathbf{1}_{X\setminus Z}, \mathbf{1}_{Y\setminus Z}\right\rangle\right| + \left|\left\langle M\mathbf{1}_Z, \mathbf{1}_{Y\setminus Z}\right\rangle\right| \\ &\quad + \left|\left\langle M\mathbf{1}_Z, \mathbf{1}_{X\setminus Z}\right\rangle\right| + 2\,|\langle M\mathbf{1}_Z, \mathbf{1}_Z\rangle| \\ &\le \delta \mathrm{vol}(V).\end{aligned}$$

Since this bound holds for any X, Y, we conclude that $\|M\|_{\mathrm{cut}} \le \delta \mathrm{vol}(V)$.

To prove (7.10) note that $\mathrm{Disc}(0.01\delta)$ implies for disjoint sets S and T

$$\left|e(S) - \frac{\mathrm{vol}(S)^2}{2\mathrm{vol}(V)}\right| \le 0.01\delta \mathrm{vol}(V), \tag{7.12}$$
$$\left|e(T) - \frac{\mathrm{vol}(T)^2}{2\mathrm{vol}(V)}\right| \le 0.01\delta \mathrm{vol}(V), \tag{7.13}$$
$$\left|e(S \cup T) - \frac{(\mathrm{vol}(S) + \mathrm{vol}(T))^2}{2\mathrm{vol}(V)}\right| \le 0.01\delta \mathrm{vol}(V). \tag{7.14}$$

If S and T are disjoint, (7.12)–(7.14) yield

$$\begin{aligned}|\langle M\mathbf{1}_S, \mathbf{1}_T\rangle| &= 2\left|e(S, T) - \frac{\mathrm{vol}(S)\mathrm{vol}(T)}{2\mathrm{vol}(V)}\right| \\ &= 2\left|e(S \cup T) - e(S) - e(T) - \frac{(\mathrm{vol}(S) + \mathrm{vol}(T))^2 - \mathrm{vol}(S)^2 - \mathrm{vol}(T)^2}{2\mathrm{vol}(V)}\right| \\ &\le 2\left|e(S) - \frac{\mathrm{vol}(S)^2}{2\mathrm{vol}(V)}\right| + 2\left|e(T) - \frac{\mathrm{vol}(T)^2}{2\mathrm{vol}(V)}\right| \\ &\quad + 2\left|e(S \cup T) - \frac{(\mathrm{vol}(S) + \mathrm{vol}(T))^2}{2\mathrm{vol}(V)}\right| \\ &\le 0.06\delta \mathrm{vol}(V),\end{aligned}$$

whence (7.10) follows. Finally, as $|\langle M\mathbf{1}_S, \mathbf{1}_S\rangle| = 2\left|e(S) - \frac{\mathrm{vol}(S)^2}{2\mathrm{vol}(V)}\right|$, the property (7.11) follows from (7.12). □

Let $D = \mathrm{diag}(d_v)_{v\in V}$ be the matrix with the vertex degrees on the diagonal and $\mathcal{M} = D^{-\frac{1}{2}} M D^{-\frac{1}{2}}$. Then the vw'th the entry of the matrix $\mathcal{M}$ is $\frac{\sqrt{d_v d_w}}{\mathrm{vol}(V)} - (d_v d_w)^{-1/2}$ if v, w are adjacent, and $\frac{\sqrt{d_v d_w}}{\mathrm{vol}(V)}$ otherwise. Establishing the following lemma is the key step.

Lemma 85. *Suppose that* $\mathrm{SDP}(M) < \epsilon^2 \mathrm{vol}(V)/64$. *Then there exists a subset* $W \subset V$ *of volume* $\mathrm{vol}(W) \geq (1-\epsilon)\cdot\mathrm{vol}(V)$ *such that* $\|\mathcal{M}_W\| < \epsilon$.

Proof. Recall that $\bar{d} = \mathrm{vol}(V)/n$. Lemma 70 implies that there is a vector $\mathbf{1} \perp z \in \mathbf{R}^V$ such that

$$\lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right] = \mathrm{SDP}(M)/n < \epsilon^2 \bar{d}/64. \tag{7.15}$$

Basically W is going to be the set of all v such that $|z_v|$ is small (and such that d_v is not too small). On the minor induced on $W \times W$ the diagonal matrix $\mathrm{diag}\binom{z}{z}$ has little effect, and thus (7.15) will imply the desired bound on $\|\mathcal{M}_W\|$. To carry out the details we need to define W precisely, bound $\|\mathcal{M}_W\|$, and prove that $\mathrm{vol}(W) \geq (1-\epsilon)\mathrm{vol}(V)$.

Let $y = D^{-1}z$ and $U = \{v \in V : d_v > \epsilon\bar{d}/8\}$. Let $y' = (y_v)_{v\in U}$ and $z' = (z_v)_{v \in U}$. Since all entries of the restricted diagonal matrix D_U exceed $\epsilon\bar{d}/8$, we have

$$\begin{aligned}
&\lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathcal{M}_U - \mathrm{diag}\begin{pmatrix} y' \\ y' \end{pmatrix}\right] \\
&= \lambda_{\max}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes D_U^{-\frac{1}{2}} \cdot \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M_U - \mathrm{diag}\begin{pmatrix} z' \\ z' \end{pmatrix}\right] \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes D_U^{-\frac{1}{2}}\right] \\
&\leq 8\left(\epsilon\bar{d}\right)^{-1} \lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M_U - \mathrm{diag}\begin{pmatrix} z' \\ z' \end{pmatrix}\right] \\
&\leq 8(\epsilon\bar{d})^{-1}\lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right] \overset{(7.15)}{<} \epsilon/8.
\end{aligned} \tag{7.16}$$

Let $W = \{v \in U : |y_v| < \epsilon/8\}$ and let $y'' = (y_v)_{v \in W}$. Then $\|\mathrm{diag}\binom{y''}{y''}\| < \epsilon/8$, because the norm of a diagonal matrix equals the largest absolute value of an entry on the diagonal. Therefore, (7.16) yields

$$\begin{aligned}
&\lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathcal{M}_W\right] \\
&\leq \lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathcal{M}_W - \mathrm{diag}\begin{pmatrix} y'' \\ y'' \end{pmatrix}\right] + \left\|\mathrm{diag}\begin{pmatrix} y'' \\ y'' \end{pmatrix}\right\| \\
&\leq \lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathcal{M}_U - \mathrm{diag}\begin{pmatrix} y' \\ y' \end{pmatrix}\right] + \left\|\mathrm{diag}\begin{pmatrix} y'' \\ y'' \end{pmatrix}\right\| \leq \epsilon/4.
\end{aligned} \tag{7.17}$$

Further, (7.17) implies that $\|\mathcal{M}_W\| < \epsilon$. To see this, let $\xi, \eta \in \mathbf{R}^W$ be a pair of unit vectors. Since $\mathcal{M}_W$ is symmetric we obtain from (7.17) and Courant-Fischer (5.5)

$$\begin{aligned}\epsilon/2 \geq 2\lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathcal{M}_W\right] &\geq \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathcal{M}_W \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} \mathcal{M}_W \eta \\ \mathcal{M}_W \xi \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle = \langle \mathcal{M}_W \eta, \xi \rangle + \langle \mathcal{M}_W \xi, \eta \rangle = 2 \langle \mathcal{M}_W \xi, \eta \rangle .\end{aligned}$$

Since this holds for any pair ξ, η, we conclude that $\|\mathcal{M}_W\| \leq \epsilon/4 < \epsilon$.

Finally, we need to show that $\mathrm{vol}(W)$ is large. To this end, we consider the set $S = \{v \in V : z_v < 0\}$. Since $\mathrm{vol}(V) = \bar{d}n \geq \bar{d}|S|$, we deduce from (7.15) and Courant-Fischer (5.5)

$$\begin{aligned}\frac{\epsilon^2 \mathrm{vol}(V)}{32} \geq \frac{\epsilon^2 \bar{d}|S|}{32} &= \frac{\epsilon^2 \bar{d}}{64} \cdot \left\| \begin{pmatrix} \mathbf{1}_S \\ \mathbf{1}_S \end{pmatrix} \right\|^2 \\ &\geq \lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right] \cdot \left\| \begin{pmatrix} \mathbf{1}_S \\ \mathbf{1}_S \end{pmatrix} \right\|^2 \\ &\geq \left\langle \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right] \cdot \begin{pmatrix} \mathbf{1}_S \\ \mathbf{1}_S \end{pmatrix}, \begin{pmatrix} \mathbf{1}_S \\ \mathbf{1}_S \end{pmatrix} \right\rangle \\ &= 2 \langle M \mathbf{1}_S, \mathbf{1}_S \rangle - 2 \sum_{v \in S} z_v. \end{aligned} \tag{7.18}$$

Further, Theorem 68 implies that

$$| \langle M \mathbf{1}_S, \mathbf{1}_S \rangle | \leq \|M\|_{\mathrm{cut}} \leq \mathrm{SDP}(M) \leq \epsilon^2 \mathrm{vol}(V)/64.$$

This combined with (7.18) and $z_v < 0$ for all $v \in S$ yields $\sum_{v \in S} |z_v| \leq \epsilon^2 \mathrm{vol}(V)/16$. Since $z \perp \mathbf{1}$, this implies $\sum_{v \in V} |z_v| \leq \epsilon^2 \mathrm{vol}(V)/8$. As $z = Dy$ and $|y_v| > \epsilon/8$ for all $v \in V \setminus W$, we obtain

$$\epsilon \mathrm{vol}(V \setminus W)/8 \leq \sum_{v \in V \setminus W} d_v |y_v| = \sum_{v \in V \setminus W} |z_v| \leq \epsilon^2 \mathrm{vol}(V)/8.$$

Hence, $\mathrm{vol}(V \setminus W) \leq \epsilon \mathrm{vol}(V)$, which implies $\mathrm{vol}(W) \geq (1 - \epsilon)\mathrm{vol}(V)$. □

Finally, we show that G satisfies ess-Eig(ϵ). Assume that G has Disc($\gamma\epsilon^2$). By Proposition 84 this implies that G satisfies Cut($100\gamma\epsilon^2$). Hence, by Theorem 68 we conclude that $\mathrm{SDP}(M) \leq \beta\epsilon^2 \mathrm{vol}(V)$ for some $0 < \beta \leq 100\theta\gamma$ and by Lemma 85 and the choice of γ there is a set W which satisfies $\mathrm{vol}(W) \geq (1 - \epsilon/10)\mathrm{vol}(V)$ and $\|\mathcal{M}_W\| < \epsilon/10$. Furthermore, $\mathcal{M}_W$ relates to the minor L_W of the Laplacian as follows. Let $\mathcal{L}_W = \boldsymbol{E} - L_W$ be the matrix whose vw'th entry is $(d_v d_w)^{-1/2}$ if $v, w \in W$ are adjacent, and 0 otherwise. Moreover, let $\Delta = (\sqrt{d_v})_{v \in W} \in \mathbf{R}^W$. Then $\mathcal{M}_W = \mathrm{vol}(V)^{-1} \Delta \Delta^T - \mathcal{L}_W$. Therefore, for all unit vectors $\xi \perp \Delta$ we have

$$|\langle L_W \xi, \xi \rangle - 1| = |\langle \mathcal{L}_W \xi, \xi \rangle| = |\langle \mathcal{M}_W \xi, \xi \rangle| \leq \|\mathcal{M}_W\| < \epsilon/10. \tag{7.19}$$

Combining (7.19) with the Courant-Fischer (5.5), we obtain

$$\lambda_2\left[L_W\right] = \max_{0 \neq \zeta \in \mathbf{R}^W} \min_{\xi \perp \zeta,\, \|\xi\|=1} \langle L_W \xi, \xi \rangle \geq \min_{\xi \perp \Delta,\, \|\xi\|=1} \langle L_W \xi, \xi \rangle \geq 1 - \epsilon. \tag{7.20}$$

To bound $\lambda_{\max}\left[L_W\right]$ as well, we need to compute $\|L_W \Delta\|^2$. To this end, recall that the row of L_W corresponding to a vertex $v \in V$ contains a one at position v. For $w \neq v$ the entry is $-(d_v d_w)^{-\frac{1}{2}}$ if v and w are adjacent, and 0 otherwise. Hence, the v-entry of the vector $L_W \Delta$ equals

$$\Delta_v - \sum_{w \in W : \{v,w\} \in E} \frac{\Delta_w}{\sqrt{d_v d_w}} = \sqrt{d_v} - \frac{e(v,W)}{\sqrt{d_v}} = \frac{d_v - e(v,W)}{\sqrt{d_v}}.$$

Since $\|\Delta\|^2 = \sum_{v \in W} d_v = \mathrm{vol}(W) \geq (1 - \epsilon/10)\mathrm{vol}(V)$, we obtain

$$\begin{aligned} \frac{\|L_W \Delta\|^2}{\|\Delta\|^2} &= \sum_{v \in W} \frac{(e(v,W) - d_v)^2}{d_v \cdot \mathrm{vol}(W)} \\ &\leq \frac{1}{1 - \epsilon/10} \sum_{v \in W} \frac{d_v - e(v,W)}{\mathrm{vol}(V)} \leq \frac{2\mathrm{vol}(V \setminus W)}{\mathrm{vol}(V)} < \epsilon/5. \end{aligned} \tag{7.21}$$

Further, decomposing any unit vector $\eta \in \mathbf{R}^W$ as $\eta = \alpha \|\Delta\|^{-1} \Delta + \beta \xi$ with a unit vector $\xi \perp \Delta$ and $\alpha^2 + \beta^2 = 1$, we get

$$\begin{aligned} \langle L_W \eta, \eta \rangle =& \left\langle L_W \left(\alpha \|\Delta\|^{-1} \Delta + \beta \xi \right), \alpha \|\Delta\|^{-1} \Delta + \beta \xi \right\rangle \\ =& \frac{\alpha^2}{\|\Delta\|^2} \cdot \langle L_W \Delta, \Delta \rangle + \frac{\alpha\beta}{\|\Delta\|} \cdot \langle L_W \Delta, \xi \rangle \\ &+ \frac{\alpha\beta}{\|\Delta\|} \cdot \langle L_W \xi, \Delta \rangle + \beta^2 \langle L_W \xi, \xi \rangle \\ =& \frac{\alpha^2}{\|\Delta\|^2} \cdot \langle L_W \Delta, \Delta \rangle + \frac{2\alpha\beta}{\|\Delta\|} \cdot \langle L_W \Delta, \xi \rangle + \beta^2 \langle L_W \xi, \xi \rangle, \end{aligned}$$

where the last step follows from the fact that L_W is symmetric. Hence, using (7.19) and (7.21), we get

$$\begin{aligned} \langle L_W \eta, \eta \rangle &\leq \frac{\alpha^2}{\|\Delta\|^2} \cdot \|L_W \Delta\| \cdot \|\Delta\| + \frac{2\alpha\beta}{\|\Delta\|} \cdot \|L_W \Delta\| \cdot \|\xi\| + \beta^2 \langle L_W \xi, \xi \rangle \\ &\leq \alpha^2 \sqrt{\epsilon/5} + 2\alpha\beta\sqrt{\epsilon/5} + \beta^2 (1 + |\langle L\xi, \xi \rangle - 1|) \\ &\leq \sqrt{\epsilon/5}(\alpha^2 + 2\alpha\beta) + \beta^2(1 + \epsilon/10) \\ &\leq 3\sqrt{\epsilon/5} \cdot |\alpha| + (1 - \alpha^2)(1 + \epsilon/10). \end{aligned}$$

By differentiating the last expression, we conclude that the maximum is attained at $\alpha = \frac{3}{2}\sqrt{\epsilon/5}/(1 + \epsilon/10)$. Plugging this value in, we obtain $\langle L_W \eta, \eta \rangle \leq 1 + \epsilon$. Hence, by

Courant-Fischer (5.5), $\lambda_{\max}[L_W] = \max_{\|\eta\|=1} \langle L_W \eta, \eta \rangle \leq 1 + \epsilon$. Thus, (7.20) shows that G has ess-Eig(ϵ).

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Erklärung

Ich erkläre hiermit, dass

- ich die vorliegende Dissertationsschrift “Extremal Hypergraph Theory and Algorithmic Regularity Lemma for Sparse Graphs” sebstständig und ohne unerlaubte Hilfe angefertigt und nur die angegebene Literatur und Hilfsmittel verwendet habe,
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Hàn, Hiệp